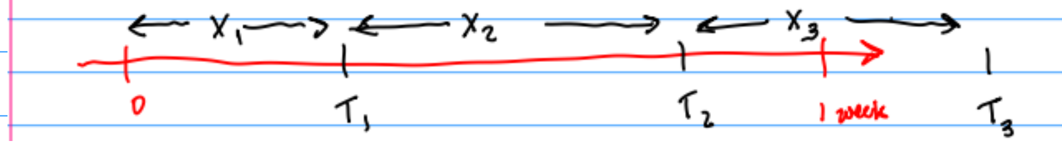
S_n is the state of inventory at the beginning of week n

D_n is the statistical model of customers trying to buy aquariums

(Same distribution every week)



recall we computed D_n using an exponential distribution (waiting time between customers)

$$P(D_n=0)=P(X_1>1) = \int_{x>1} f(x)\,dx = \int_1^\infty \lambda e^{-\lambda x}\,dx$$

for any n

$$= \int_1^\infty e^{-x}\,dx = -e^{-x}\Big|_1^\infty = e^{-1} \approx$$

```
julia> exp(-1.0)
0.36787944117144233
```

$$P(D>0) = 1 - P(D=0) = 1 - e^{-1} =$$

```
julia> 1-exp(-1)
0.6321205588285577
```

$$P(D=1) = P(X_1+X_2>1) - P(X_1>1) = 2e^{-1} - e^{-1} = e^{-1}$$

```
julia> exp(-1.0)
0.36787944117144233
```

$$P(D>1) = 1 - P(D=1) - P(D=0) = 1 - 2e^{-1} =$$

```
julia> 1-2*exp(-1)
0.26424111765711533
```

$$P(D=2) = P(X_1+X_2+X_3>1) - P(X_1+X_2>1)$$

$$= \frac{1}{2}e^{-1} + \cancel{P(X_1+X_2>1)} - \cancel{P(X_1+X_2>1)} =$$

```
julia> 1/2*exp(-1)
0.18393972058572117
```

$$P(D>2) = 1 - \left(1+1+\frac{1}{2}\right)e^{-1} =$$

```
julia> 1-(1+1+1/2)*exp(-1)
0.08030139707139416
```

pattern

$$P(D=n) = \frac{1}{n!} e^{-1}$$

$$P(D=3) = P(T_4 > 1) - P(T_3 > 1) = \frac{1}{3!} e^{-1} = \frac{1}{6} e^{-1}$$

```
julia> 1/6*exp(-1)
0.06131324019524038
```

$$\mathbb{P}(D>3) = 1 - P(D=0) - P(D=1) - P(D=2) - P(D=3)$$
$$= 1 - \left(1 + 1 + \frac{1}{2} + \frac{1}{6}\right) e^{-1} =$$

```
julia> 1-(1+1+1/2+1/6)*exp(-1)
0.018988156876153815
```

Recall that $\pi_n = (\mathbb{P}(S_n=1), \mathbb{P}(S_n=2), \mathbb{P}(S_n=3))$ row vector

and $\pi_{n+1} = \pi_n P$ where $p_{ij}^{n} = \mathbb{P}(S_{n+1}=j \mid S_n=i)$

look like depends on n ... but markov property means it doesn't.

$$\lim_{n\to\infty} \pi_n =$$

```
julia> println(pn)
[0.28471134182344127 0.26313999184626332 0.4521486663302953]
```

Idea Compute $P(D_n > S_n)$ to see what percentage of weeks there was more demand than inventory.

How about on average? $\displaystyle \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(D_n > S_n)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(D_n > S_n) = \lim_{n\to\infty} P(D_n > S_n)$$

because everything converges to the equilibrium.

$$P(D_n > S_n) = P(D_n > S_n \mid S_n = 1)\,P(S_n = 1) + P(D_n > S_n \mid S_n = 2)\,P(S_n = 2) + P(D_n > S_n \mid S_n = 3)\,P(S_n = 3)$$

$$= P(D_n > 1)P(S_n = 1) + P(D_n > 2)P(S_n = 2) + P(D_n > 3)P(S_n = 3)$$

$$= \sum_{i=1}^{3} P(D_n > i)\,P(S_n = i)$$

$$= \big(P(D_n > 1), P(D_n > 2), P(D_n > 3)\big) \cdot \big(P(S_n = 1), P(S_n = 2), P(S_n = 3)\big)$$

don't depend on n because we assume customer demand follow same distribution each week.

recall

$$\pi_n = \big(\mathbb{P}(S_n = 1), \mathbb{P}(S_n = 2), \mathbb{P}(S_n = 3)\big)$$

Therefore

$$P(D_n > S_n) = \big(0.264, 0.080, .0190\big) \cdot \pi_n$$

$$\lim_{n \to \infty} P(D_n > S_n) = \lim_{n \to \infty} \big(0.264, 0.080, .0190\big) \cdot \pi_n$$

$$= \lim_{n \to \infty} \big(0.264, 0.080, .0190\big) \cdot (0.285, 0.263, 0.452)$$

=

```
julia> [0.264,0.080,0.0190]'*[0.285,0.263,0.452]
0.104868
```

from the note

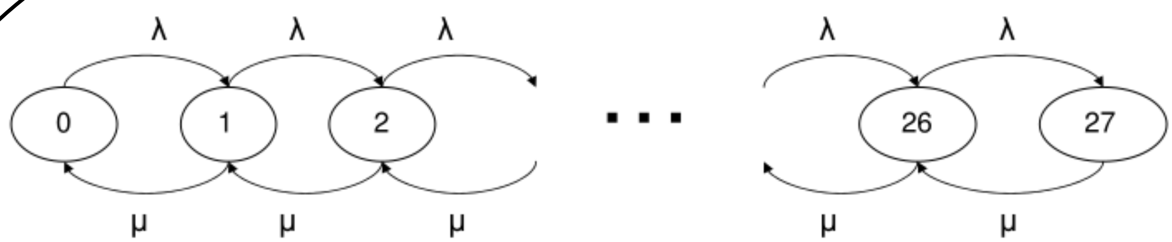
$$\Pr\{D_n > S_n\} = \sum_{i=1}^{3} \Pr\{D_n > S_n \mid X_n = i\}\Pr\{X_n = i\}$$

$$= (.264)(.285) + (.080)(.263) + (.019)(.452) = .105$$

That was Markov Chain (discrete week by week) process that models inventory (other things) with a transition diagram.

Markov process in the continuous in time version.

rate diagram



Let $P_n(t)$ be the probability that there are n forklifts in the shop at time t.

Namely $P_n(t) = \mathbb{P}(X_t = n)$,

X_t is number of fork lifts in the shop at time t.

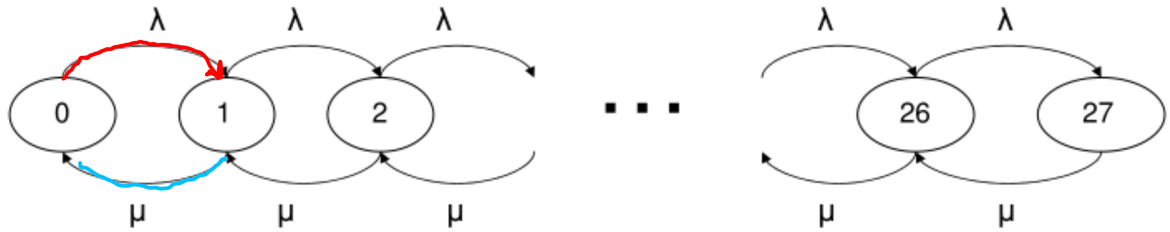
Idea is a continuous time version of $\pi_{n+1} = \pi_n P$.

$P(t) = (P_0(t), P_1(t), \dots, P_n(t)) \in \mathbb{R}^{1\times 28}$ row vector...

$$\frac{dP(t)}{dt} = P(t)A$$ ← flow matrix

diagonal so the rows add to zero... this is "conservation of forklifts" in the flow problem..

$$A = \begin{array}{c c} & \begin{array}{cccccc} & 2 & & & & \end{array} \\ 1 & \begin{bmatrix} -\lambda & \lambda & & & 0 \\ \mu & -\mu-\lambda & \ddots & & \\ & \mu & \ddots & \lambda & \\ & & \ddots & -\mu-\lambda & \lambda \\ 0 & & & \mu & -\mu \end{bmatrix} \end{array}$$



λ
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27
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