1. Consider the continuous-time whale problem given by

$$\frac{dB}{dt} = 0.05B\Big(1 - \frac{B}{150000}\Big) - \alpha BF$$

$$\frac{dF}{dt} = 0.08F\Big(1 - \frac{F}{400000}\Big) - \alpha BF$$

We will use simulation to explore the fractal limit sets for the discrete approximations of the continuous system using Euler's method for different step sizes h.

(i) Let $X = (B, F)$ and write the above system as $dX/dt = G(X)$. What is $G(X)$?

The above system can be written as $dX/dt = G(X)$ where

$$X = \begin{bmatrix} B \\ F \end{bmatrix} \qquad \text{and} \qquad G(X) = \begin{bmatrix} 0.05B(1 - B/150000) - \alpha BF \\ 0.08F(1 - F/400000) - \alpha BF \end{bmatrix}.$$

(ii) Suppose the initial whale populations are $B_0 = 5000$ blue whales and $F_0 = 70000$ fin whales. Let $\alpha = 10^{-8}$ and use a computer implementation of the Euler method

$$X_{n+1} = X_n + hG(X_n) \qquad \text{with} \qquad h = 32$$

to plot the population of fin whales for the times $t_n = hn$ where $n = 0, 1, \ldots, 50$. This may be done in Julia as

```
Xn=X0; Xs=[X0]; h=32
for n=1:50
    Xn=Xn+h*G(Xn...)
    push!(Xs,Xn)
end
scatter(h*(0:50),last.(Xs),
    xlabel="years",ylabel="Fin whales",label=false)
```

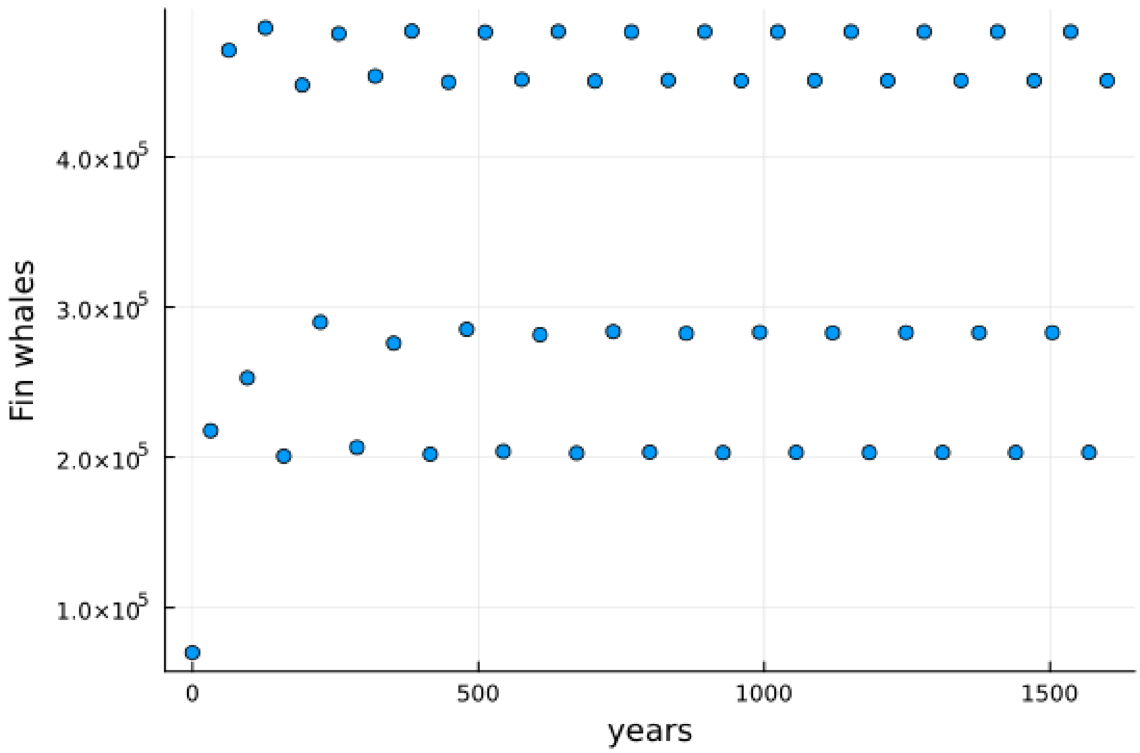
In [1]:
```
G(B,F)=[0.05*B*(1-B/150000)-alpha*B*F,
        0.08*F*(1-F/400000)-alpha*B*F]
alpha=1e-8
B0=5000.0; F0=70000.0
X0=[B0,F0]
```

Out[1]:
```
2-element Vector{Float64}:
  5000.0
 70000.0
```

In [2]:
```julia
using Plots
```

In [3]:
```julia
Xn=X0; Xs=[X0]; h=32
for n=1:50
    Xn=Xn+h*G(Xn...)
    push!(Xs,Xn)
end
scatter(h*(0:50),last.(Xs),
    xlabel="years",ylabel="Fin whales",label=false)
```

Out[3]:



(iii) Plot blue whales B_n versus fin whales F_n for $n = 100, 101, \ldots, 1000$. This may be done in Julia as

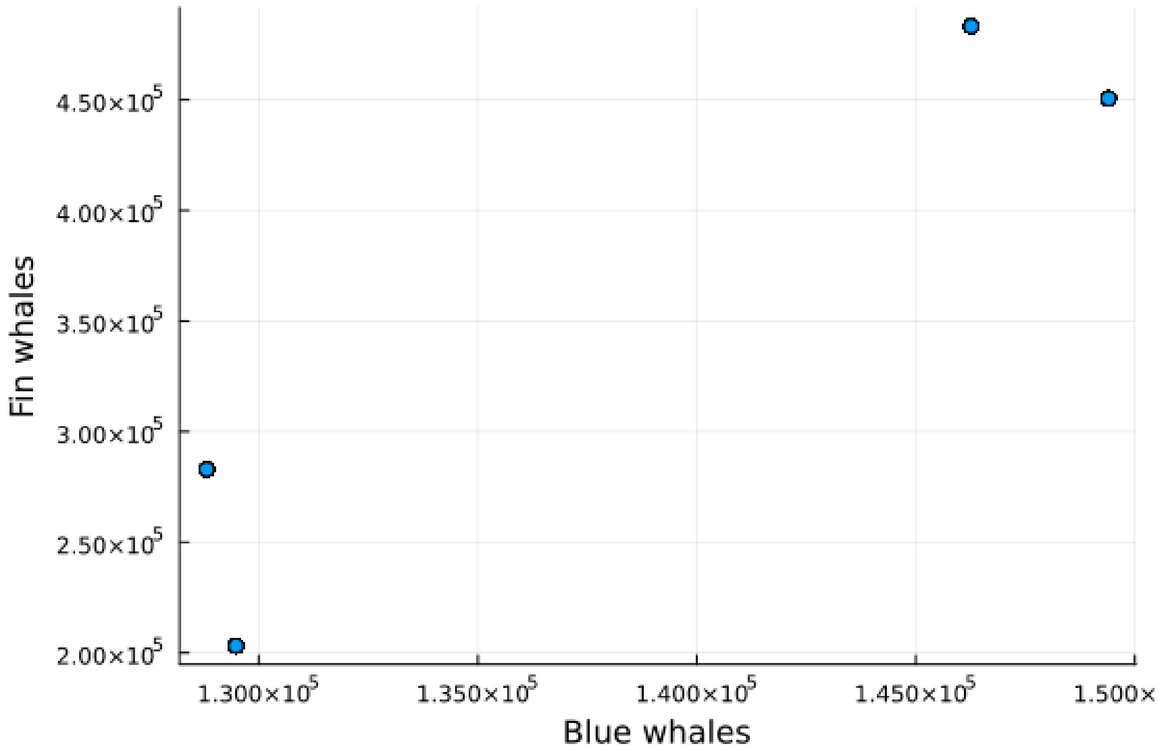
```julia
Xn=X0; Xs=[X0]; h=32
for n=1:1000
    Xn=Xn+h*G(Xn...)
    push!(Xs,Xn)
end
scatter(first.(Xs[101:end]),last.(Xs[101:end]),
    xlabel="Blue whales",ylabel="Fin whales",label=false)
```

Your graph should show the limit set consisting of four points.

In [4]:
```julia
Xn=X0; Xs=[X0]; h=32
for n=1:1000
    Xn=Xn+h*G(Xn...)
    push!(Xs,Xn)
end
```

```
scatter(first.(Xs[101:end]),last.(Xs[101:end]),
    xlabel="Blue whales",ylabel="Fin whales",label=false)
```

Out[4]:



(iv) Repeat the simulation of part (iii) for step sizes of $h = 33, 34, \ldots, 37$. For each case plot the limit set. How does the limit set change as the step size increases?

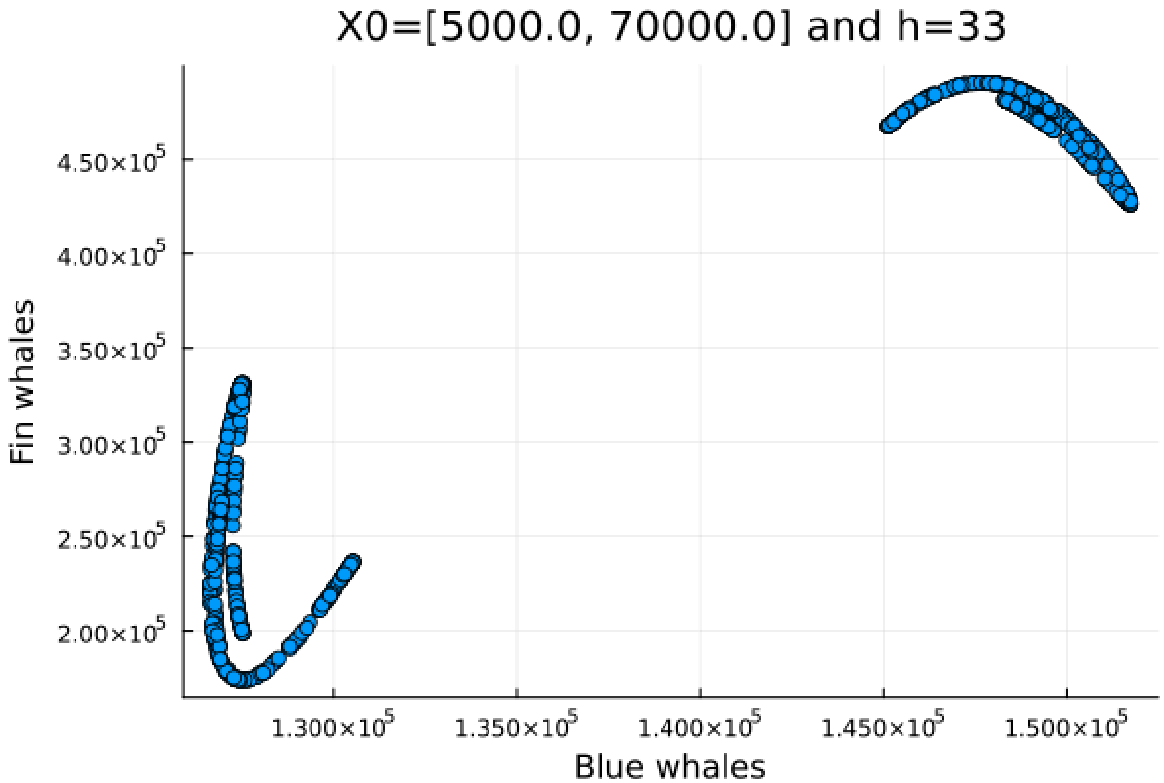
In [5]:

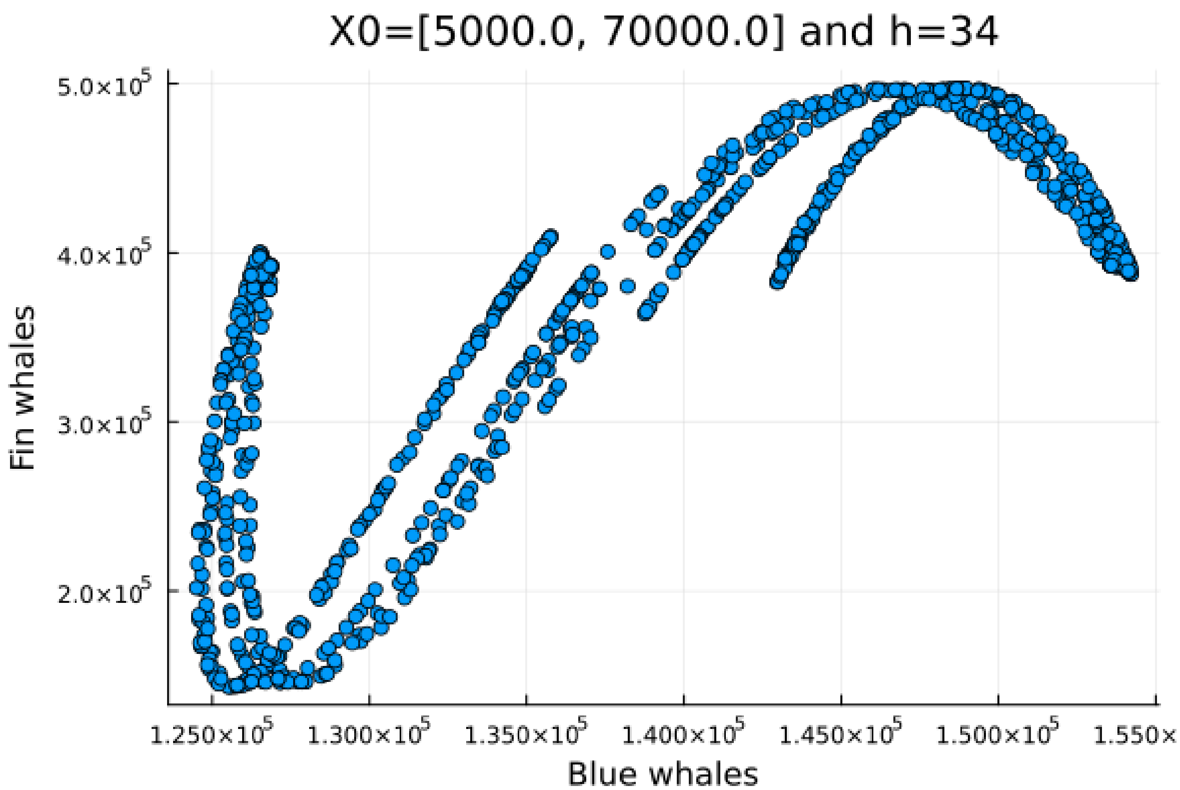
```
function mkplot(h)
    Xn=X0; Xs=[X0]
    for n=1:1000
        Xn=Xn+h*G(Xn...)
        push!(Xs,Xn)
    end
    scatter(first.(Xs[101:end]),last.(Xs[101:end]),
        title="X0=$X0 and h=$h",
        xlabel="Blue whales",ylabel="Fin whales",label=false)
end
```

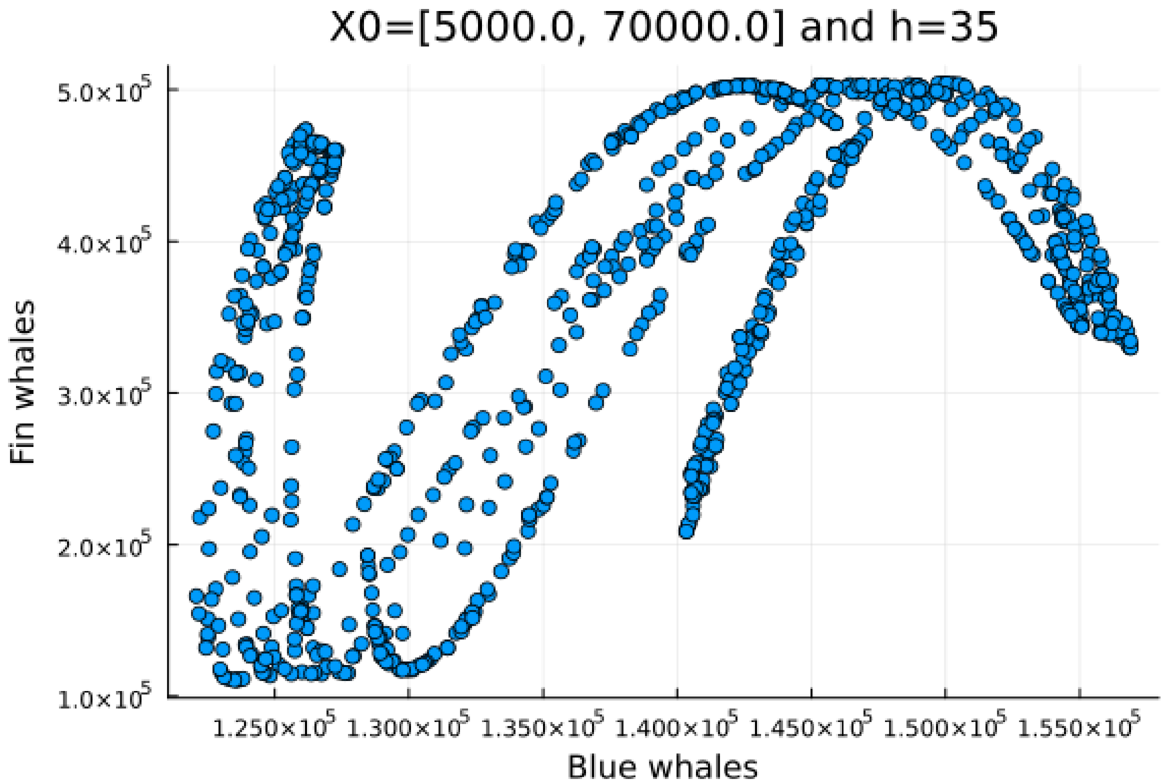
Out[5]: mkplot (generic function with 1 method)

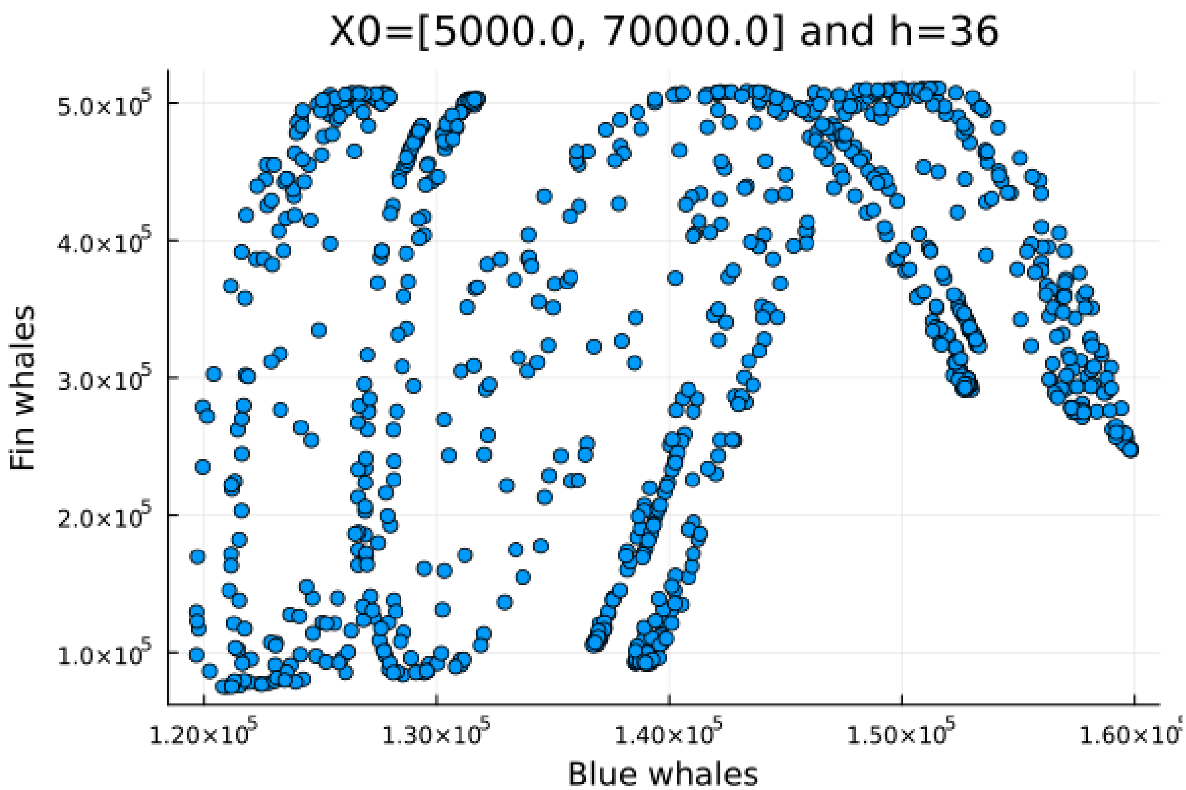
In [6]:

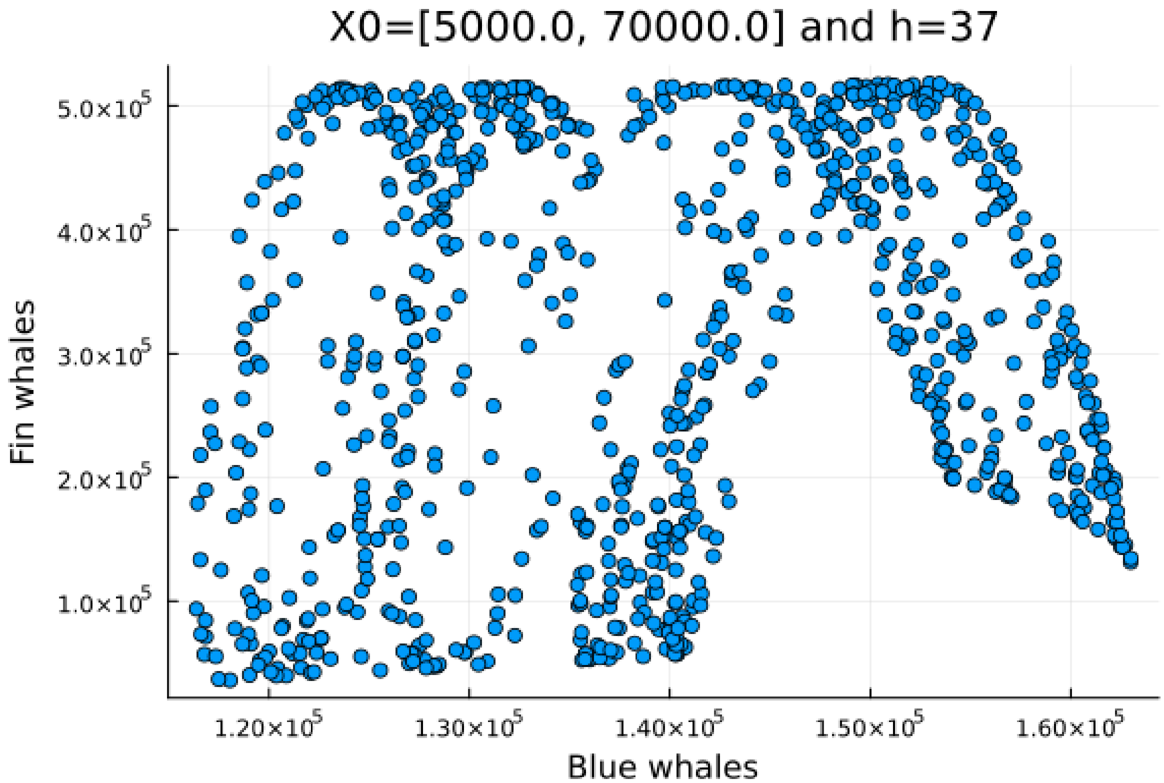
```
for h=33:37
    println()
    display(mkplot(h))
end
```


X0=[5000.0, 70000.0] and h=33
Fin whales
Blue whales
4.50×10^5
4.00×10^5
3.50×10^5
3.00×10^5
2.50×10^5
2.00×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5


X0=[5000.0, 70000.0] and h=34
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.250×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5


X0=[5000.0, 70000.0] and h=35
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.0×10^5
1.250×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5
1.550×10^5


X0=[5000.0, 70000.0] and h=36
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.0×10^5
1.20×10^5
1.30×10^5
1.40×10^5
1.50×10^5



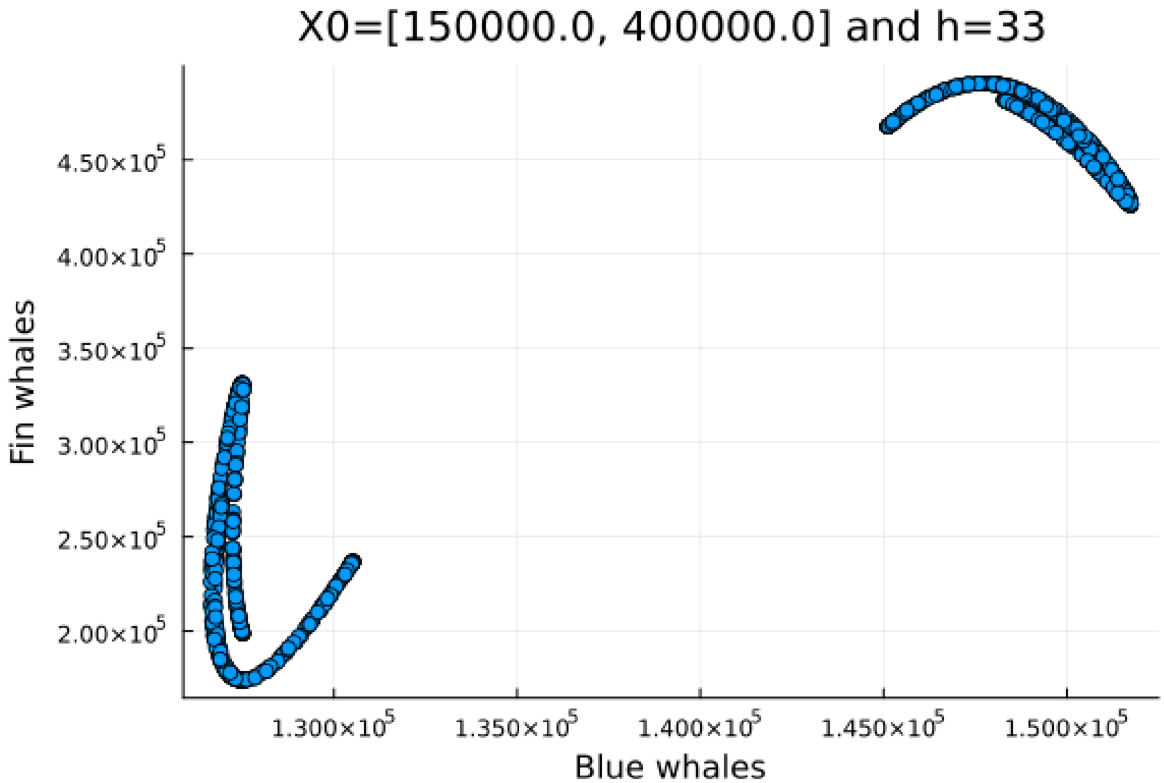
As the step size h increases the limit set gets more and more complicated. Note, however, that the number of fin and blue whales varies over a similar range in each case.

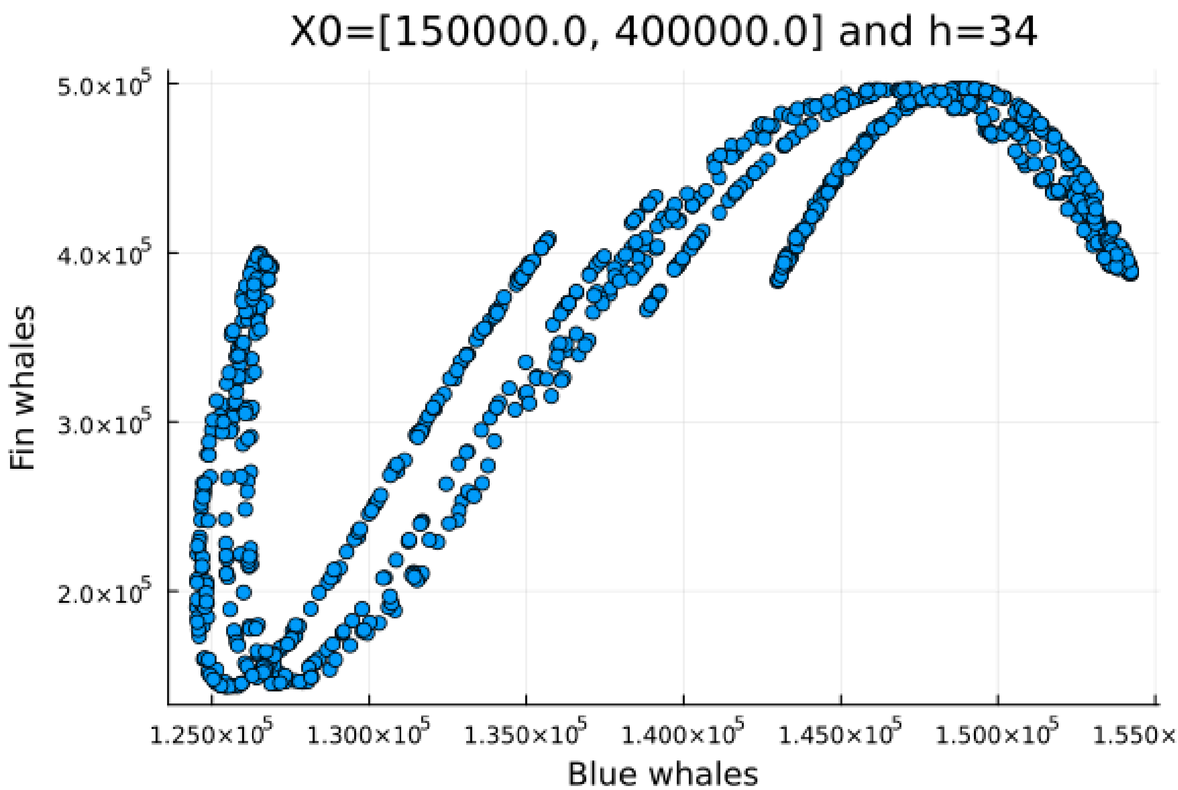
(iv) Repeat part (iv) for the initial condition $B_0 = 150000$ and $F_0 = 400000$. Does the limit set depend on the initial condition?

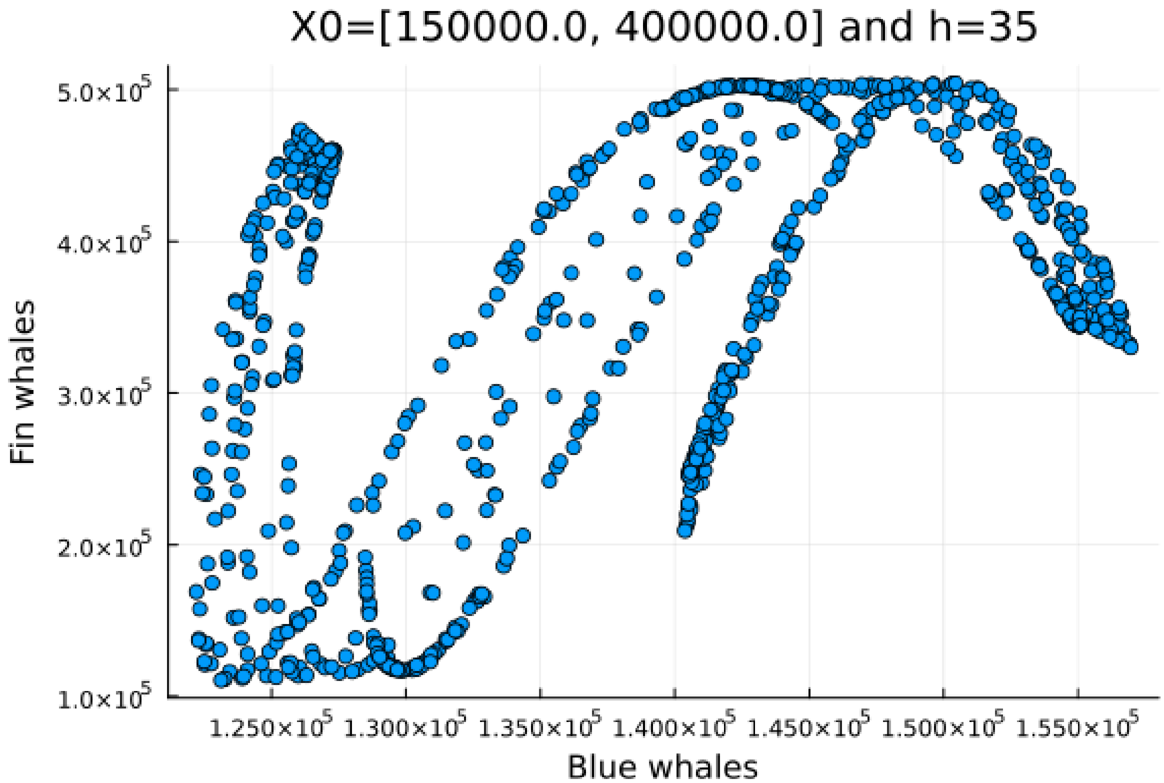
In [7]:
```
B0=150000.0; F0=400000.0
X0=[B0,F0]
```

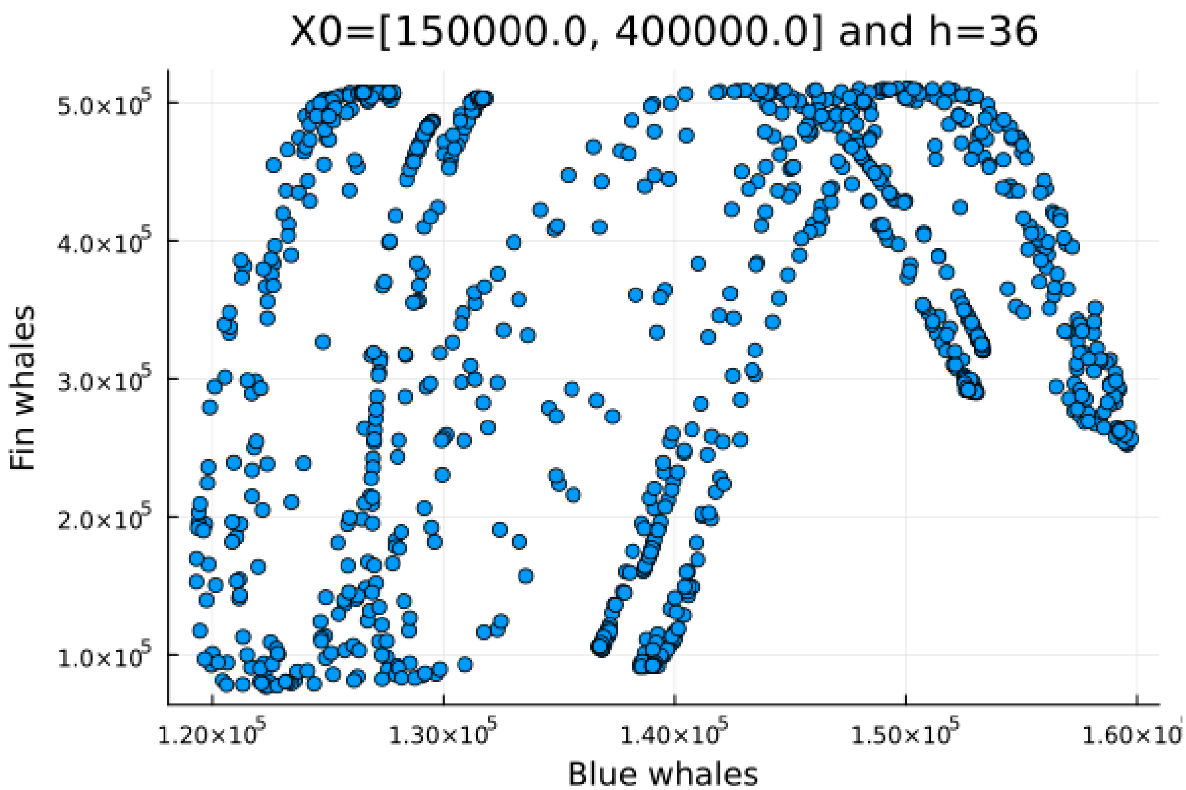
Out[7]:
```
2-element Vector{Float64}:
 150000.0
 400000.0
```

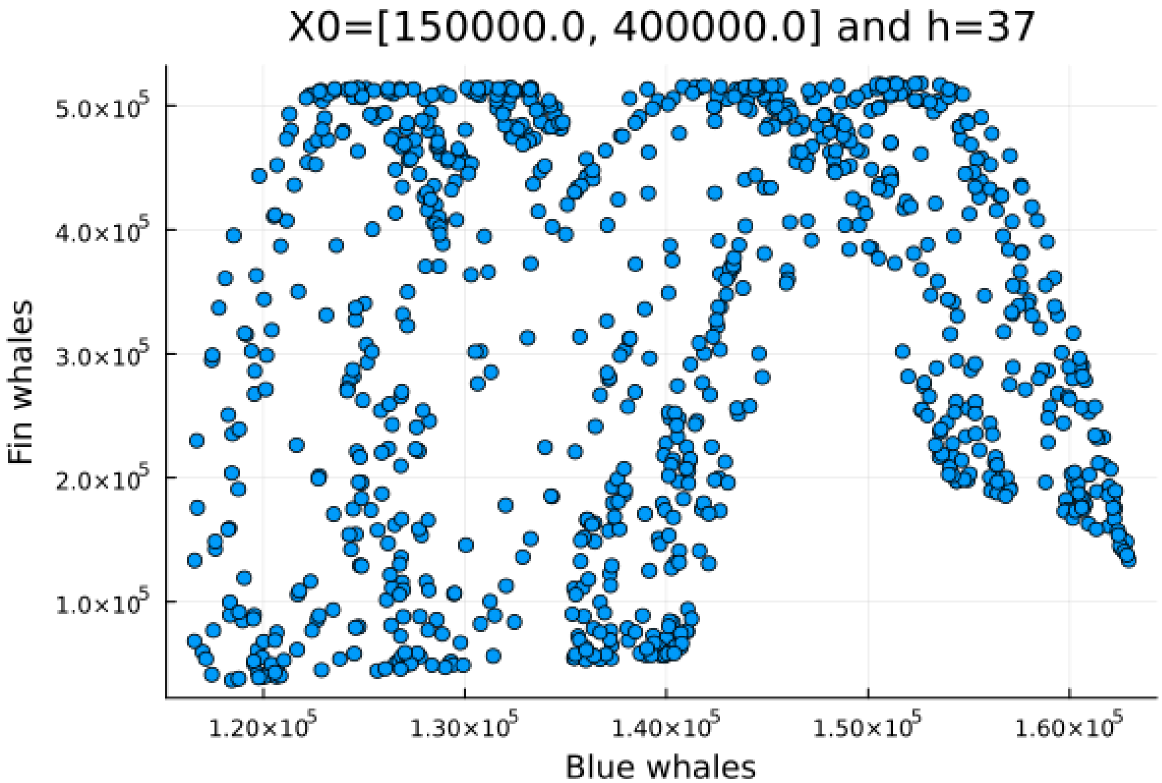
In [8]:
```
for h=33:37
    println()
    display(mkplot(h))
end
```


X0=[150000.0, 400000.0] and h=33
Fin whales
Blue whales
4.50×10^5
4.00×10^5
3.50×10^5
3.00×10^5
2.50×10^5
2.00×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5


X0=[150000.0, 400000.0] and h=34
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.250×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5


X0=[150000.0, 400000.0] and h=35
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.0×10^5
1.250×10^5
1.300×10^5
1.350×10^5
1.400×10^5
1.450×10^5
1.500×10^5
1.550×10^5


X0=[150000.0, 400000.0] and h=36
Fin whales
Blue whales
5.0×10^5
4.0×10^5
3.0×10^5
2.0×10^5
1.0×10^5
1.20×10^5
1.30×10^5
1.40×10^5
1.50×10^5



The limit set is visually indistinguishable when comparing the choice of $X_0 = (5000, 70000)$ to $X_0 = (150000, 400000)$. Therefore it appears the limit set does not depend on the initial condition.

```
In [ ]:
```