2. An electronics manufacturer produces a variety of diodes. Quality control engineers attempt to insure that faulty diodes will be detected in the factory before they are shipped. It is estimated that 0.2% of the diodes produced will be faulty. It is possible to test each diode individually. It is also possible to place a number of diodes in series and test the entire group. If this test fails, it means that one or more of the diodes in that group are faulty. The estimated testing cost is 4 cents for a single diode, and $3+n$ cents for a group of $n>1$ diodes. If a group test fails, then each diode in the group must be retested individually.

(i) Derive the average cost function $A(n)$ that should be minimized in this case.

Let D be a random variable representing a diode. Thus,

$$D=\begin{cases}\text{faulty} & \text{with probability } q\\ \text{good} & \text{with probability } p=1-q.\end{cases}$$

Let B be a random variable representing a test batch of n diodes. Thus,

$$B=\begin{cases}\text{all good} & \text{with probability } p^n\\ \text{at least one bad} & \text{with probability } 1-p^n.\end{cases}$$

Let C represent the cost of testing a batch of diodes. Thus,

$$C=\begin{cases}4 & \text{if } n=1\\ 3+n & \text{if } n>1 \text{ and } B=\text{all good}\\ 3+n+4n & \text{if } n>1 \text{ and } B=\text{at least one bad}.\end{cases}$$

Assuming $n>1$ it follows that the expected cost per batch is

$$\begin{aligned}\mathbf{E}[C]&=(3+n)\mathbf{P}(B=\text{all good})+(3+n+4n)\mathbf{P}(B=\text{at least one bad})\\ &=(3+n)p^n+\big((3+n)+4n\big)(1-p^n)=3+5n-4np^n.\end{aligned}$$

Therefore, the average per-diode cost function that should be minimized is

$$A(n)=\mathbf{E}[C/n]=\frac{\mathbf{E}[C]}{n}=\frac{3+5n-4np^n}{n}=\frac{3}{n}+5-4p^n.$$

(ii) Find the minimum value of $A(n)$ over the set $n=1,2,3,\ldots$.

In [1]:
```
q=0.002; p=1-q
log(4/3.5)/log(1/p)
```

Out[1]:
```
66.69890833843708
```

Note that $p^n<3.5/4$ provided $n\log p<\log(3.5/4)$ or equivalently when

$$n > \frac{\log(3.5/4)}{\log p} = \frac{\log(4/3.5)}{\log(1/p)} \approx 66.7.$$

It follows that $n \geq 67$ implies

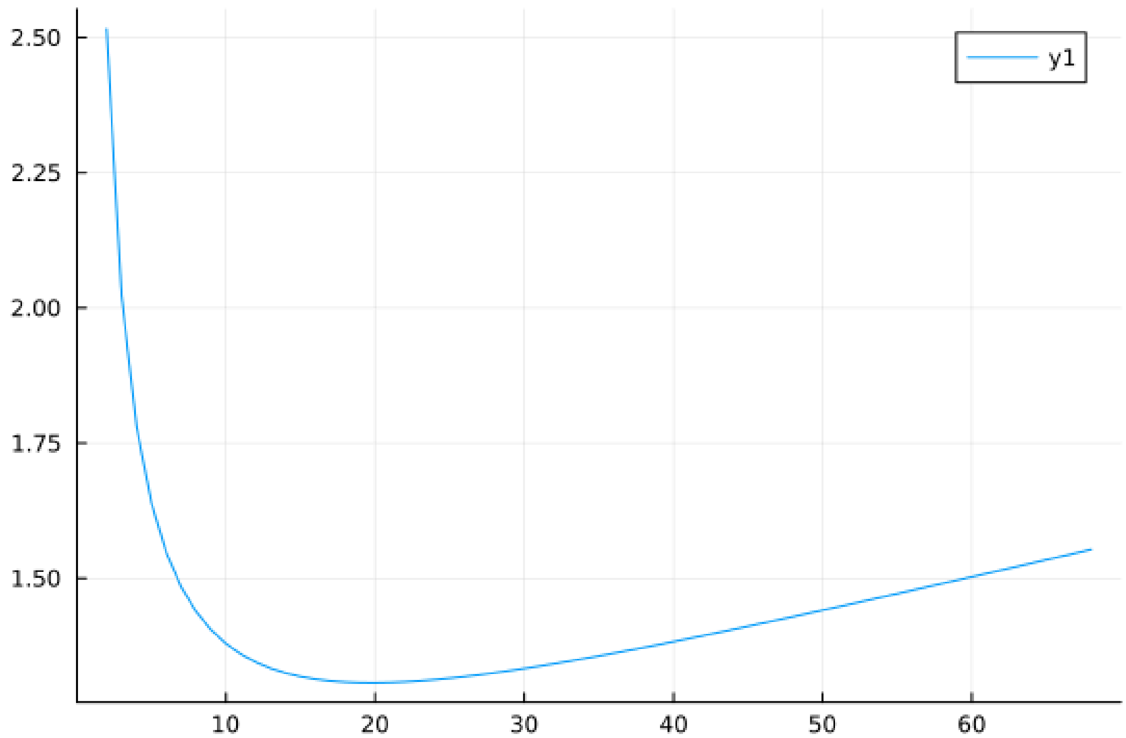
$$A(n) = \frac{3}{n} + 5 - 4p^n > 5 - 4(3.5/4) = 5 - 3.5 = 1.5.$$

Plotting $A(n)$ over the rangle $[2, 68]$ yields

In [2]:
```
using Plots
```

In [3]:
```
A(n)=3/n+5-4*p^n
plot(A,2:68)
```

Out[3]:



From the graph we see that the minimum of $A(n)$ is somewhere between 10 and 30. Therefore, we test $A(n)$ for each integer between these values and find the minimum.

In [4]:
```
T=vcat([[n A(n)] for n=10:30]...)
```

Out[4]:
```
21×2 Matrix{Float64}:
 10.0  1.37928
 11.0  1.35985
 12.0  1.34495
 13.0  1.33353
 14.0  1.32484
 15.0  1.31833
 16.0  1.3136
 17.0  1.31032
 18.0  1.30824
 19.0  1.30719
 20.0  1.307
 21.0  1.30754
 22.0  1.30872
 23.0  1.31044
 24.0  1.31265
 25.0  1.31527
 26.0  1.31827
 27.0  1.32159
 28.0  1.3252
 29.0  1.32907
 30.0  1.33317
```

In [5]:
```
nmin=Int(T[argmin(T[:,2]),1])
```

Out[5]:
```
20
```

In [6]:
```
Amin=A(nmin)
```

Out[6]:
```
1.3069961718946295
```

It follows that the minimum occurs at $n = 20$ and is $A(20) \approx 1.3069961718946295$.

(iii) Let q be the probability that a diode is faulty. Compute the relative sensitivity $S(A, q)$ evaluated at $q = 0.002$.

Assume for infinitesimal small changes the integer n where the minimum occurs doesn't change. Then

In [7]:
```
using Symbolics
D(f,x)=expand_derivatives(Differential(x)(f))
```

Out[7]:
```
D (generic function with 1 method)
```

In [8]:
```
@variables p,q
```

Out[8]:
```
2-element Vector{Num}:
 p
 q
```

In [9]:
```
Aq=substitute(A(nmin),p=>1-q)
```

Out[9]: 5.15 - 4((1 - q)^20)

In [10]:
```
DAdq=D(Aq,q)
SAq=substitute(q/Amin*DAdq,q=>0.002)
```

Out[10]: 0.11784901352849506

It follows that the relative sensitivity is

$$S(A,q) = \frac{q}{A}\frac{dA}{dq}\Big|_{q=0.002} \approx 0.11784901352849506.$$

In [ ]: