3. Consider an RLC circuit consisting of a capacitor, an active resistor with v-i characteristic $f(x) = x^5 - \rho x$ and an inductor. Assume $\rho > 0$, $L = 1$ and $C = 1$.

(i) Write the system of differential equations. Plot the direction field for $i_C \in [-2, 2]$ and $v_C \in [-2, 2]$ when $\rho = 1$.

Since the circuit is a closed loop than the current through each component is the same. Thus, $i_R = i_L = i_C$. By Kirchhoff's law $v_R + v_L + v_C = 0$. Finally for each component we have

$$v_R = f(i_R), \qquad L\frac{di_L}{dt} = v_L \qquad \text{and} \qquad C\frac{dv_C}{dt} = i_C.$$

Substituting yields

$$v_L = -v_R - v_C = -f(i_R) - v_C = -f(i_C) - v_C = -i_C^5 + \rho i_C - v_C.$$

Therefore, the system becomes

$$L\frac{di_C}{dt} = -i_C^5 + \rho i_C - v_C$$

$$C\frac{dv_C}{dt} = i_C.$$

In vector form $dX/dt = F(X)$ where

$$X = \begin{bmatrix} i_C \\ v_C \end{bmatrix} \qquad \text{and} \qquad F(X) = \begin{bmatrix} (-i_C^5 + \rho i_C - v_C)/L \\ i_C/C \end{bmatrix}.$$

In [1]:
```
F(X)=[(-X[1]^5+rho*X[1]-X[2])/L,X[1]/C]
```

Out[1]:
```
F (generic function with 1 method)
```

In [2]:
```
using Plots, LinearAlgebra
```

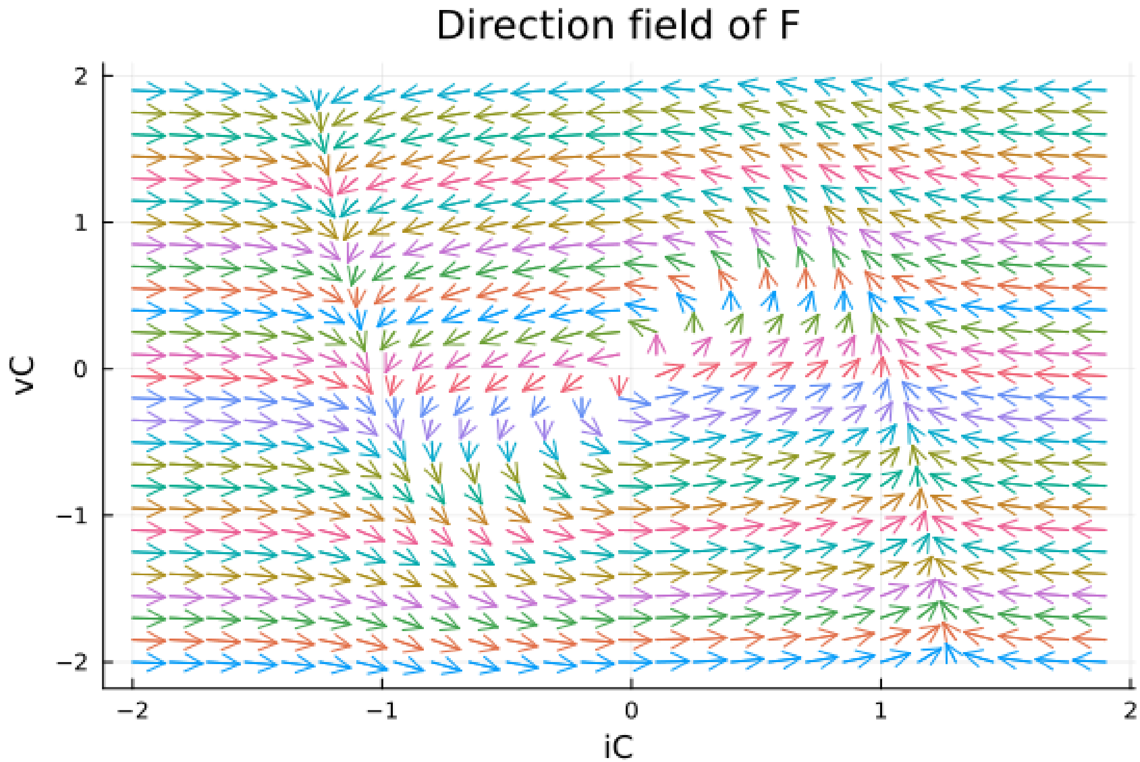
In [3]:
```
rho=1; L=1; C=1
```

Out[3]:
```
1
```

In [4]:
```
is=-2:0.15:2
vs=-2:0.15:2
quiver(is*ones(length(vs))',ones(length(is))*vs',
```

```
    quiver=(x,y)->F([x,y])/norm(F([x,y]))*0.13,
    title="Direction field of F",xlabel="iC",ylabel="vC")
```

Out[4]:



(ii) Linearize the dynamical system in the vicinity of the equilibrium when $i_C = 0$ and $v_C = 0$. Find the eigenvalues $\lambda_1 = \alpha_1 + i\beta_1$ and $\lambda_2 = \alpha_2 + i\beta_2$ as a function of ρ. Plot the real parts $\alpha_1(\rho)$ and $\alpha_2(\rho)$ for $\rho \in [0, 4]$.

In [5]:
```
using Symbolics
@variables x,y,rho
```

Out[5]:
```
3-element Vector{Num}:
   x
   y
 rho
```

In [6]:
```
DF=Symbolics.jacobian(F([x,y]),[x,y])
```

Out[6]:
```
2×2 Matrix{Num}:
 rho - 5(x^4)  -1
           1    0
```

In [7]:
```
Arho=substitute(DF,[x=>0,y=>0])
```

Out[7]:
```
2×2 Matrix{Num}:
 rho  -1
   1   0
```

The eigenvalues are roots of the characteristic polynomial

$$\chi(\lambda) = \det \begin{bmatrix} \rho - \lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda(\lambda - \rho) + 1 = \lambda^2 - \rho\lambda + 1.$$

By the quadratic formula $\chi(\lambda) = 0$ setting $a = 1$, $b = -\rho$ and $c = 1$ to obtain

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\rho \pm \sqrt{\rho^2 - 4}}{2} = \begin{cases} \dfrac{\rho \pm \sqrt{\rho^2 - 4}}{2} & \text{when } \rho > 2 \\ \dfrac{\rho \pm i\sqrt{4 - \rho^2}}{2} & \text{when } \rho < 2. \end{cases}$$

It follows that $\lambda_1 = \alpha_1 + i\beta_1$ where

$$\alpha_1 = \begin{cases} \dfrac{\rho - \sqrt{\rho^2 - 4}}{2} & \text{when } \rho > 2 \\ \dfrac{\rho}{2} & \text{when } \rho < 2 \end{cases} \qquad \text{and} \qquad \beta_1 = \begin{cases} 0 & \text{whe} \\ \dfrac{-\sqrt{4 - \rho^2}}{2} & \text{whe} \end{cases}$$

and $\lambda_2 = \alpha_2 + i\beta_2$ where

$$\alpha_2 = \begin{cases} \dfrac{\rho + \sqrt{\rho^2 - 4}}{2} & \text{when } \rho > 2 \\ \dfrac{\rho}{2} & \text{when } \rho < 2 \end{cases} \qquad \text{and} \qquad \beta_2 = \begin{cases} 0 & \text{when} \\ \dfrac{\sqrt{4 - \rho^2}}{2} & \text{when} \end{cases}$$

In [8]:
```
Arhos="A(rho)="*string(Symbolics.toexpr(Arho))
eval(Meta.parse(Arhos))
```

Out[8]:
```
A (generic function with 1 method)
```

In [9]:
```
eigvals(A(1.0))
```

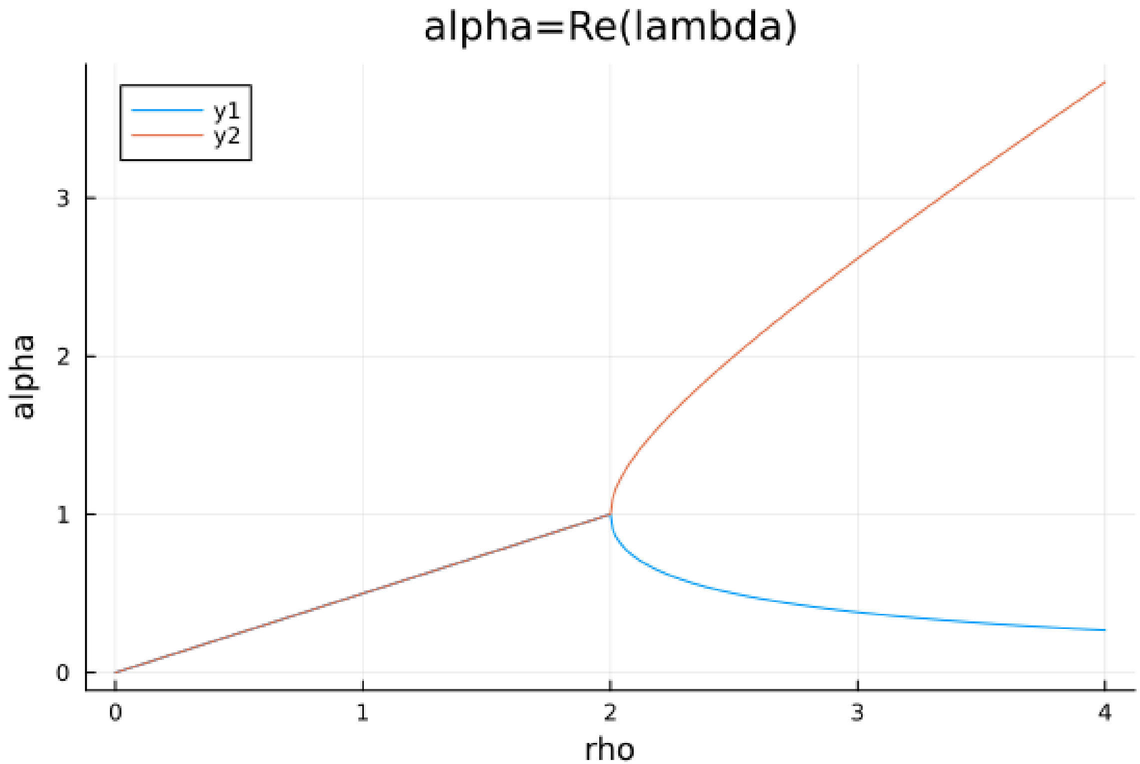
Out[9]:
```
2-element Vector{ComplexF64}:
 0.5 - 0.8660254037844385im
 0.5 + 0.8660254037844385im
```

In [10]:
```
alpha1(rho)=real(eigvals(A(rho))[1])
alpha2(rho)=real(eigvals(A(rho))[2])
```

Out[10]:
```
alpha2 (generic function with 1 method)
```

In [11]:
```
plot([rho->alpha1(rho),rho->alpha2(rho)],0:0.01:4,
    xlabel="rho",ylabel="alpha",title="alpha=Re(lambda)")
```

Out[11]:



(iii) Determine the stability of the equilibrium for $\rho \in [0, 4]$.

The real parts of the eigenvalues are positive for all values $\rho > 0$. That means the equilibrium at $(0, 0)$ is unstable for all values of $\rho > 0$. When $\rho = 0$ the eigenvalues have zero real part. In this case the linearized equation does not determine the stability of the original system.

(iv) Plot the direction field of the linear system for $i_C \in [-2, 2]$ and $v_C \in [-2, 2]$ when $\rho = 1$. How does the linear direction field compare with the one in (i)?

In [12]:

```
A1=A(1.0)
```

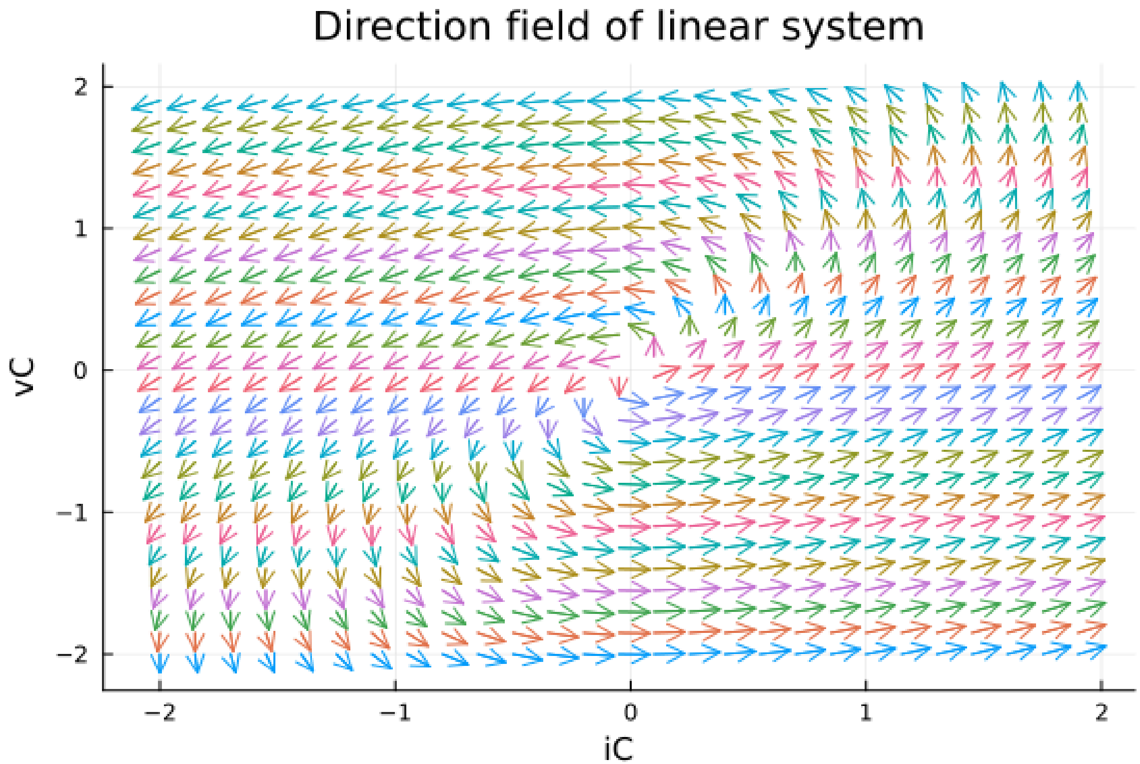
Out[12]:

```
2×2 Matrix{Float64}:
 1.0  -1.0
 1.0   0.0
```

In [13]:

```
quiver(is*ones(length(vs))',ones(length(is))*vs',
    quiver=(x,y)->A1*[x,y]/norm(A1*[x,y])*0.13,
    title="Direction field of linear system",xlabel="iC",ylabel="vC")
```

Out[13]:



The direction field for the linear system looks system to the original system when $|i_C| \ll 1$ and $|v_C| \ll 1$. However, around $|i_C| \approx 1$ the arrows change direction and are completely reversed after that. This is the effect of the non-linearity that appears to give rise to the oscillator similar to the Van der Pol oscillator.

In [ ]: