Math 420/620 Final Question 2

2. In an unmanaged tract of forest area, hardwood and softwood trees compete for the available land and water according to the the model

$$\frac{dH}{dt} = r_1 H\Big(1 - \frac{\gamma S + H}{K_1}\Big)$$
$$\frac{dS}{dt} = r_2 S\Big(1 - \frac{S + \gamma H}{K_2}\Big).$$

Assume that hardwoods grow at a rate of $r = 0.10$ per year and softwoods at a rate of $r_2 = 0.25$ per year. An acre of forest land can support about $K_1 = 10000$ tons of hardwoods or $K_2 = 6000$ tons of softwoods. Further suppose the factor $\gamma = 3/4$.

(i) Plot the direction field for this model. Indicate the location of each equilibrium in the state space.

First write the forest model in vector form $dX/dt = F(X)$ by setting

$$X = \begin{bmatrix} H \\ S \end{bmatrix} \qquad \text{and} \qquad F(H,S) = \begin{bmatrix} r_1 H\big(1 - (\gamma S + H)/K_1\big) \\ r_2 S\big(1 - (S + \gamma H)/K_2\big) \end{bmatrix}$$

In [1]:
```
r1=0.1; r2=0.25; K1=10000; K2=6000; gamma=3/4
```

Out[1]:
```
0.75
```

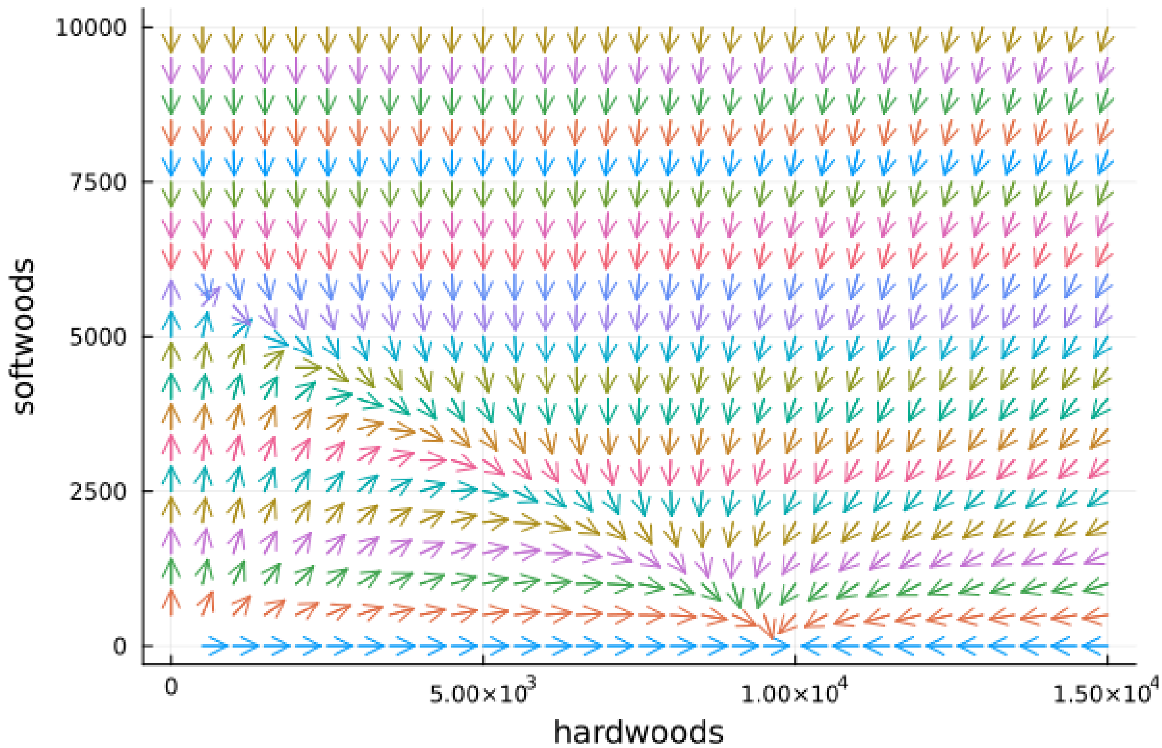
In [2]:
```
F(H,S)=[r1*H*(1-(gamma*S+H)/K1),
        r2*S*(1-(S+gamma*H)/K2)]
```

Out[2]:
```
F (generic function with 1 method)
```

In [3]:
```
using Plots, LinearAlgebra
```

In [4]:
```
Hs=0:500:15000
Bs=0:500:10000
quiver(
    Hs*ones(length(Bs))',ones(length(Hs))*Bs',
    quiver=(H,S)->F(H,S)/norm(F(H,S))*400,
    xlabel="hardwoods",ylabel="softwoods"
)
```

Out[4]:



Now find the fixed points when $F(H,S)=0$ or equivalently when

$$r_1 H\Big(1-\frac{\gamma S+H}{K_1}\Big)=0 \qquad \text{and} \qquad r_2 S\Big(1-\frac{S+\gamma H}{K_2}\Big)=0.$$

We shall work in cases.

Case $H=0$. Then the second equation reduces to

$$r_2 S\Big(1-\frac{S}{K_2}\Big)=0$$

and consequently either $S=0$ or $S=K_2$. Therefore $(0,0)$ and $(0,K_2)$ are fixed points.

Case $H\neq 0$. Then dividing the first equation yields

$$\Big(1-\frac{\gamma S+H}{K_1}\Big)=0$$

which implies $\gamma S+H=K_1$ or that $S=(K_1-H)/\gamma$. Now substitute this into the second equation to obtain

$$r_2\frac{K_1-H}{\gamma}\Big(1-\frac{(K_1-H)/\gamma+\gamma H}{K_2}\Big)=0.$$

It follows that either

$$H=K_1 \qquad \text{or} \qquad (K_1-H)/\gamma+\gamma H=K_2.$$

If $H = K_1$ then $S = (K_1 - H)/\gamma = 0$ and so $(K_1, 0)$ is a fixed point.

If $(K_1 - H)/\gamma + \gamma H = K_2$ then

$$\frac{K_1}{\gamma} - K_2 = H\Big(\frac{1}{\gamma} - \gamma\Big) \qquad \text{implies} \qquad H = \frac{\gamma}{1-\gamma^2}\Big(\frac{K_1}{\gamma} - K_2\Big)$$

and consequently

$$S = \frac{K_1 - H}{\gamma} = \frac{K_1}{\gamma} - \frac{1}{1-\gamma^2}\Big(\frac{K_1}{\gamma} - K_2\Big) = \frac{\gamma}{1-\gamma^2}\Big(\frac{K_2}{\gamma} - K_1\Big).$$

It follows there is another fixed point at

$$\Big(\frac{K_1 - \gamma K_2}{1-\gamma^2}, \frac{K_2 - \gamma K_1}{1-\gamma^2}\Big).$$

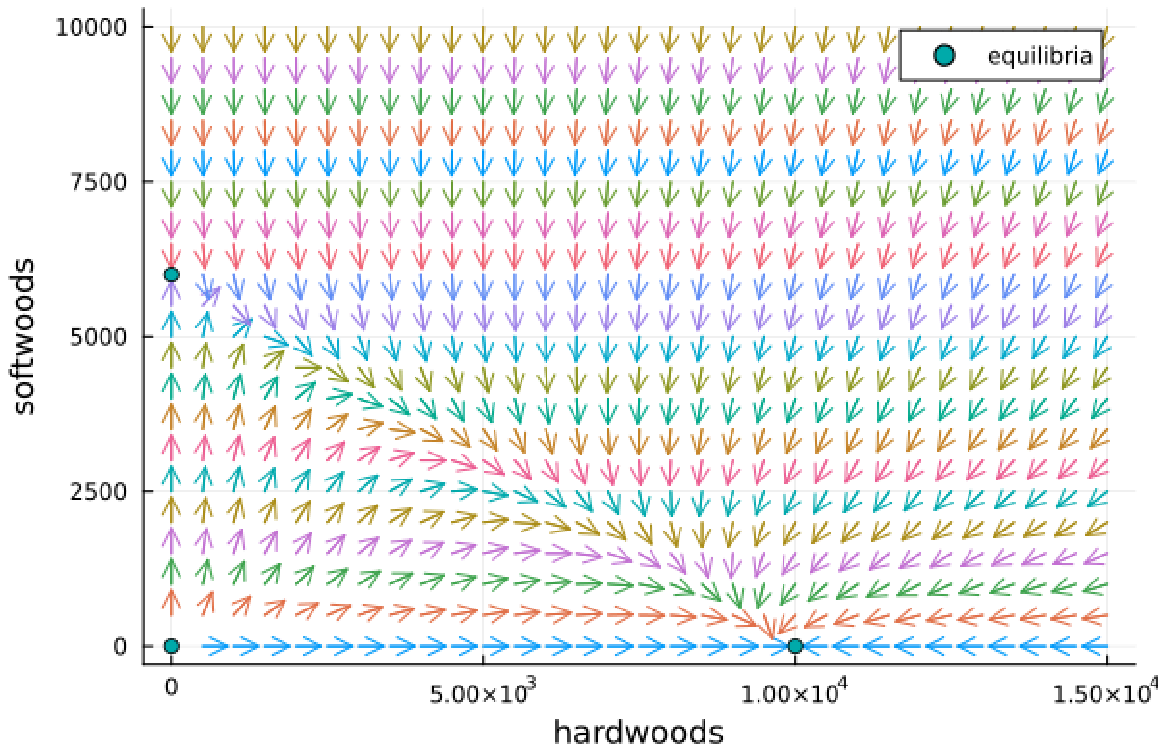
Note, however, by symmetry that at least coordinate of the above fixed point must be negative. Since negative amounts of trees don't make sense, we don't consider it here.

In [5]:
```
p1=[0,0]
p2=[0,K2]
p3=[K1,0]
p4=[K1-gamma*K2,K2-gamma*K1]/(1-gamma^2)
fp=[p1,p2,p3,p4]
```

Out[5]:
```
4-element Vector{Vector{Float64}}:
 [0.0, 0.0]
 [0.0, 6000.0]
 [10000.0, 0.0]
 [12571.42857142857, -3428.5714285714284]
```

In [6]:
```
scatter!(first.(fp[1:3]),last.(fp[1:3]),label="equilibria")
```

Out[6]:



(ii) For each equilibrium point, determine the linear system that approximates the behavior of the original dynamical system in the neighborhood of the equilibrium point. Write the general solution to this linear system. Plot the direction field for the linear model.

Let the fixed points be

$$P_1 = (0, 0), \qquad P_2 = (0, K_2) \qquad \text{and} \qquad P_3 = (K_1, 0).$$

Given a dynamical system

$$\frac{dX}{dt} = F(X)$$

the linearization

$$F_L(X) = F(P_i) + DF(P_i)(X - P_i)$$

reduces to

$$F_L(X) = DF(P_i)(X - P_i)$$

when P_i is a fixed point since $F(P_i) = 0$.

We compute $A_i = DF(P_i)$ for each of the three fixed points found in part (i), set $Y = X - P_i$, and then solve the resulting linear system

$$\frac{dY}{dt} = A_i Y.$$

It follows that $X(t) = Y(t) + P_i$.

To begin recall that

$$\frac{dH}{dt} = r_1 H\Big(1 - \frac{\gamma S + H}{K_1}\Big)$$
$$\frac{dS}{dt} = r_2 S\Big(1 - \frac{S + \gamma H}{K_2}\Big).$$

Consequently,

$$DF(H,S) = \begin{bmatrix} r_1\big(1 - \frac{\gamma S + 2H}{K_1}\big) & -r_1 \frac{\gamma H}{K_1} \\ -r_2 \frac{\gamma S}{K_2} & r_2\big(1 - \frac{2S + \gamma H}{K_2}\big) \end{bmatrix}$$

For $P_1 = (0,0)$ we have

$$A_1 = DF(H,S)\Big|_{(H,S)=(0,0)} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

Since A_1 is diagonal, the eigenvectors are

$$K_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad K_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 0.1$ and $\lambda_2 = 0.25$. The general solution is

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0.1t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{0.25t}.$$

In [7]:
```
A1=[r1 0; 0 r2]
```

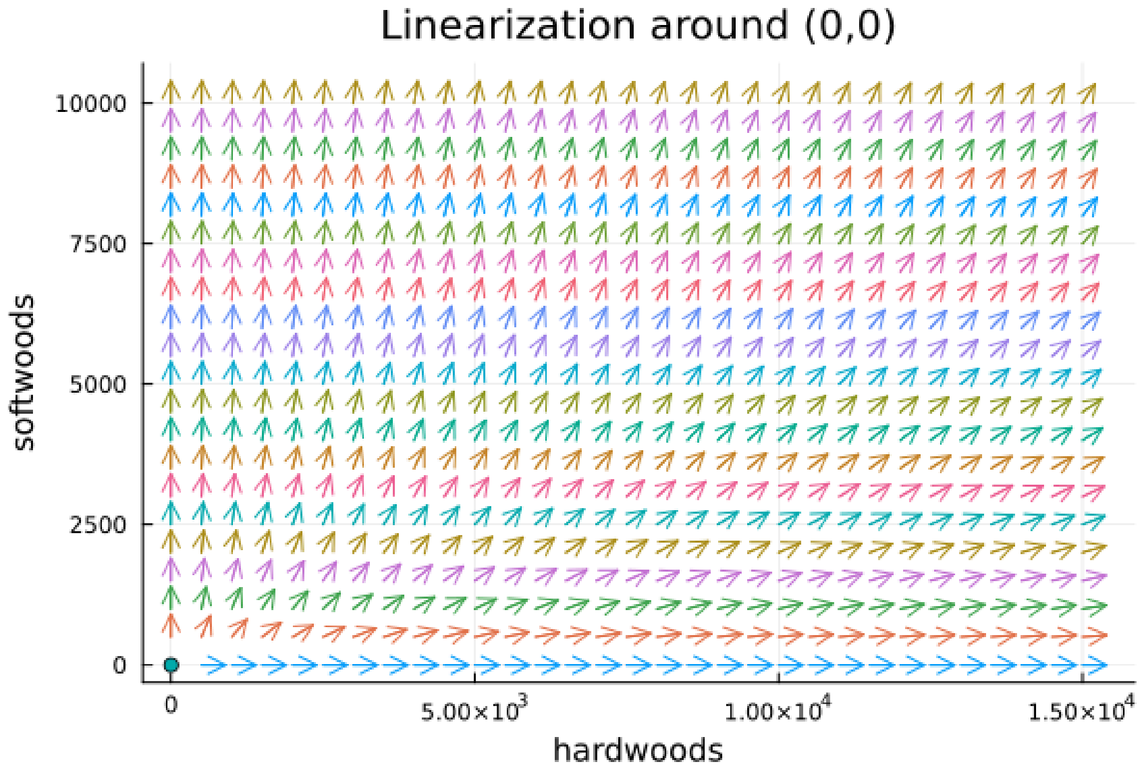
Out[7]:
```
2×2 Matrix{Float64}:
 0.1  0.0
 0.0  0.25
```

Since both eigenvalues are positive this is an unstable source fixed point.

In [8]:
```
quiver(
    Hs*ones(length(Bs))',ones(length(Hs))*Bs',
    quiver=(H,S)->A1*[H,S]/norm(A1*[H,S])*400,
    title="Linearization around (0,0)",
    xlabel="hardwoods",ylabel="softwoods"
```

```
)
scatter!([p1[1]],[p1[2]],legend=:false)
```

Out[8]:



For $P_2 = (0, K_2)$ we have

$$A_2 = DF(H,S)\Big|_{(H,S)=(0,K_2)} = \begin{bmatrix} r_1\left(1-\gamma\frac{K_2}{K_1}\right) & 0 \\ -r_2\gamma & -r_2 \end{bmatrix}$$

In [9]:

```
A2=[r1*(1-gamma*K2/K1) 0; -r2*gamma -r2]
```

Out[9]:

```
2×2 Matrix{Float64}:
  0.055    0.0
 -0.1875  -0.25
```

In [10]:

```
eigvecs(A2)
```

Out[10]:

```
2×2 Matrix{Float64}:
 0.0   0.851898
 1.0  -0.523708
```

In [11]:

```
eigvals(A2)
```

Out[11]:

```
2-element Vector{Float64}:
 -0.25
  0.055000000000000001
```

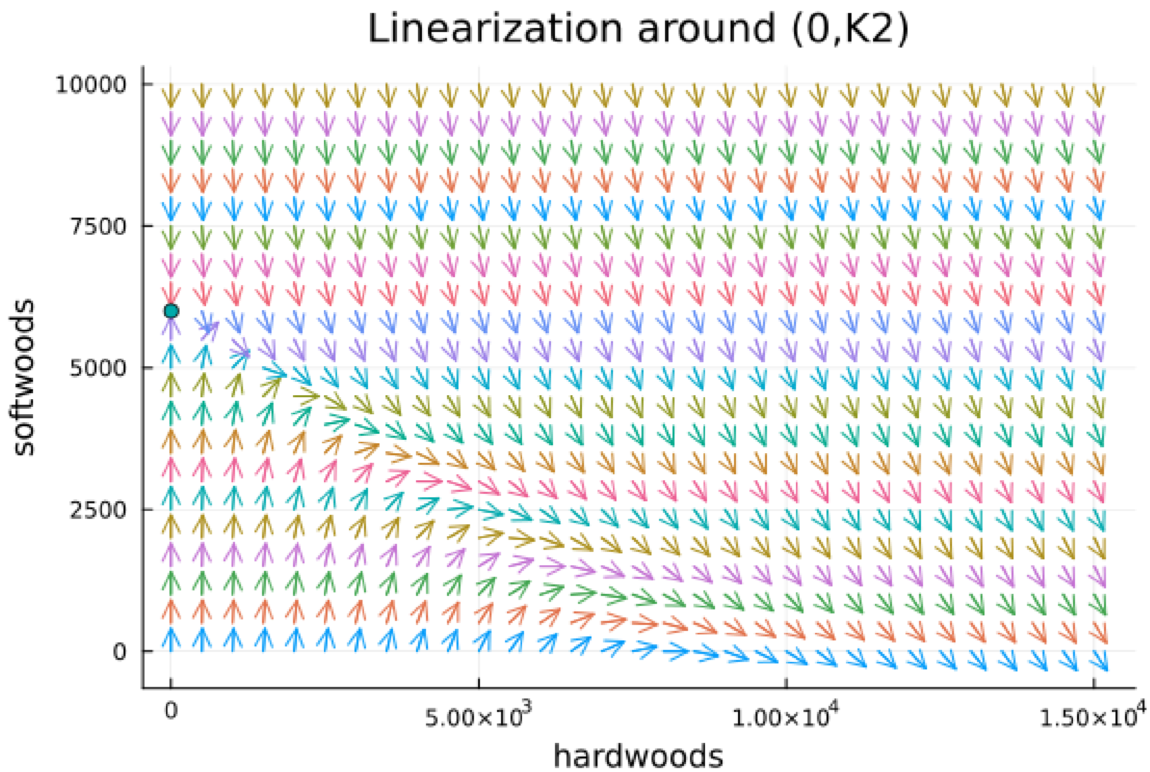
Since one eigenvalue is positive and the other is negative this is an unstable saddle fixed point.

The general solution is

$$X(t) = \begin{bmatrix} 0 \\ 6000 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-0.25t} + c_2 \begin{bmatrix} 0.851898 \\ -0.523708 \end{bmatrix} e^{0.055t}.$$

In [12]:
```
F2(H,S)=A2*([H,S]-[0,K2])
quiver(
    Hs*ones(length(Bs))',ones(length(Hs))*Bs',
    quiver=(H,S)->F2(H,S)/norm(F2(H,S))*400,
    title="Linearization around (0,K2)",
    xlabel="hardwoods",ylabel="softwoods"
)
scatter!([p2[1]],[p2[2]],legend=:false)
```

Out[12]:



For $P_3 = (K_1, 0)$ we have

$$A_3 = DF(H,S)\Big|_{(H,S)=(K_1,0)} = \begin{bmatrix} -r_1 & -r_2\gamma \\ 0 & r_2\left(1 - \gamma\frac{K_1}{K_2}\right) \end{bmatrix}$$

In [13]:
```
A3=[-r1 -r2*gamma; 0 r2*(1-gamma*K1/K2)]
```

Out[13]:
```
2×2 Matrix{Float64}:
 -0.1  -0.1875
  0.0  -0.0625
```

In [14]:
```
eigvecs(A3)
```

Out[14]:
```
2×2 Matrix{Float64}:
 1.0  -0.980581
 0.0   0.196116
```

Since both eigenvalues are negative, this is an stable sink fixed point.

In [15]:
```
eigvals(A3)
```

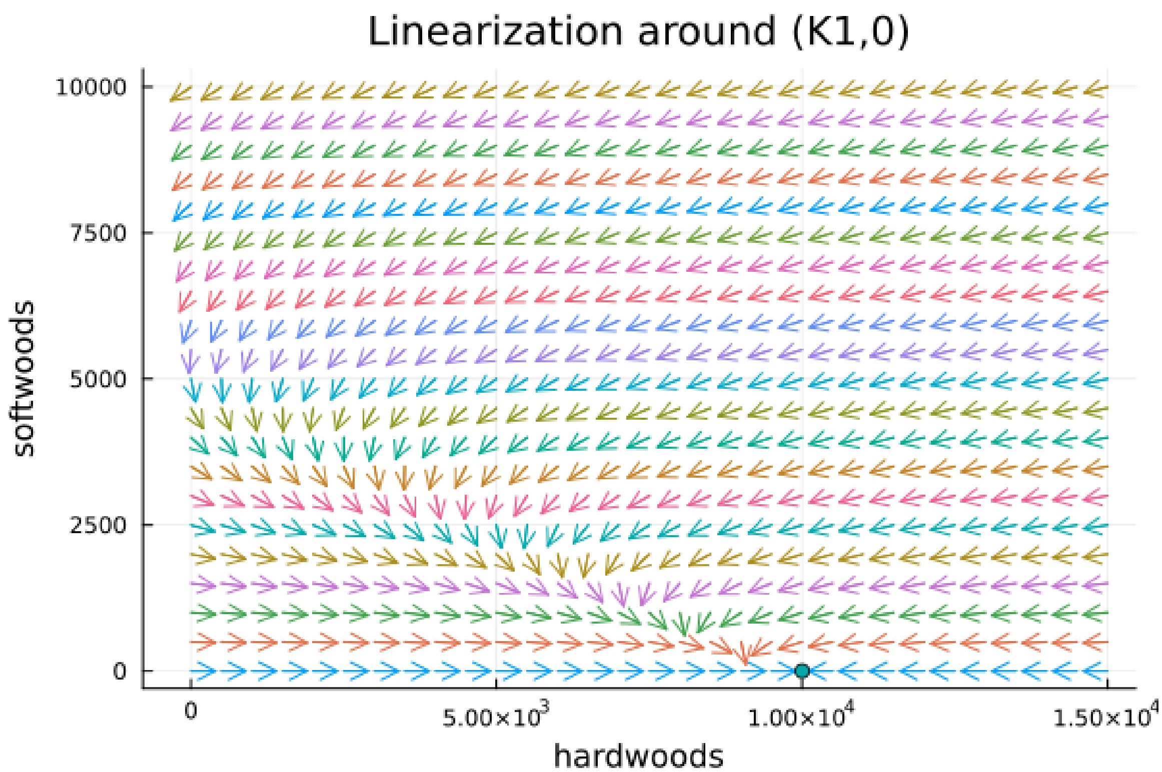
Out[15]:
```
2-element Vector{Float64}:
 -0.1
 -0.0625
```

The general solution is

$$X(t) = \begin{bmatrix} 10000 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-0.1t} + c_2 \begin{bmatrix} -0.980581 \\ 0.196116 \end{bmatrix} e^{-0.0625t}.$$

In [16]:
```
F3(H,S)=A3*([H,S]-[K1,0])
quiver(
    Hs*ones(length(Bs))',ones(length(Hs))*Bs',
    quiver=(H,S)->F3(H,S)/norm(F3(H,S))*400,
    title="Linearization around (K1,0)",
    xlabel="hardwoods",ylabel="softwoods"
)
scatter!([p3[1]],[p3[2]],legend=:false)
```

Out[16]:



(iii) Suppose that a small number of hardwood trees is introduced into a mature stand of softwood trees. What does our model predict about the future of this forest?

If $H = 0$ then a mature softwood forest would be at the equilibrium $(0, K_2)$. That is, a mature softwood forest would have 6000 tons per acre of softwood trees and no hardwood trees. Introducting a small number of hardwood trees implies an initial condition $X_0 = (\varepsilon, K_2)$ for some $\varepsilon > 0$.

Now, since this the fixed point at $(0, K_2)$ is an unstable saddle and the stable direction of the linearized system is in the vertical direction while we have perturbed in the horizontal, then (ε, K_2) is not on the stable manifold of and will consequently be attracted to the stable fixed point at $(K_1, 0)$.

The conclusion is that the future as $t \to \infty$ of the forest will be all hardwood trees. In particular, there will be 10000 tons of hardwood trees per acre and no software trees.

Use Euler's method

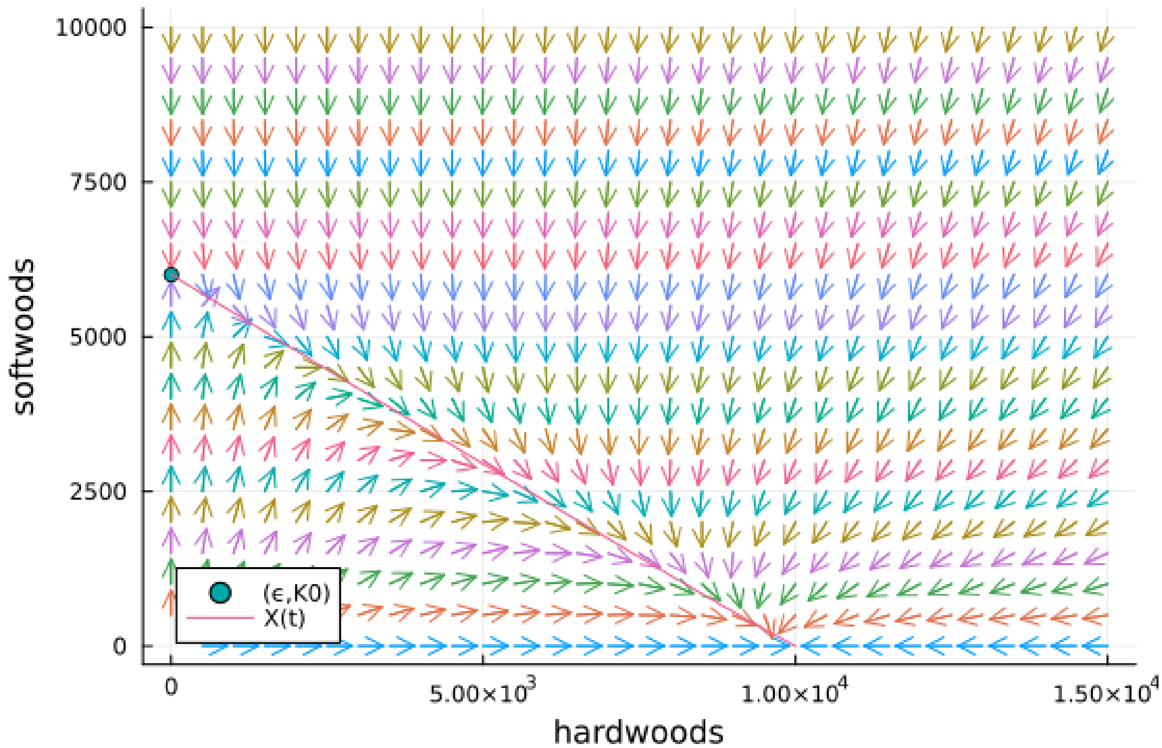
$$X_{n+1} = X_n + hF(X_n) \qquad \text{where} \qquad X_0 = (\varepsilon, K_2)$$

to approximate the evolution of the mature software forest to the hardwood forest.

In [17]:
```
epsilon=0.01
h=0.1; N=4000
Xn=[epsilon,K2]
Xs=[Xn]
for n=1:N
    Xn+=h*F(Xn...)
    push!(Xs,Xn)
end
```

In [18]:
```
quiver(
    Hs*ones(length(Bs))',ones(length(Hs))*Bs',
    quiver=(H,S)->F(H,S)/norm(F(H,S))*400,
    xlabel="hardwoods",ylabel="softwoods"
)
scatter!([epsilon],[K2],label="(ϵ,K0)")
plot!(first.(Xs),last.(Xs),label="X(t)")
```

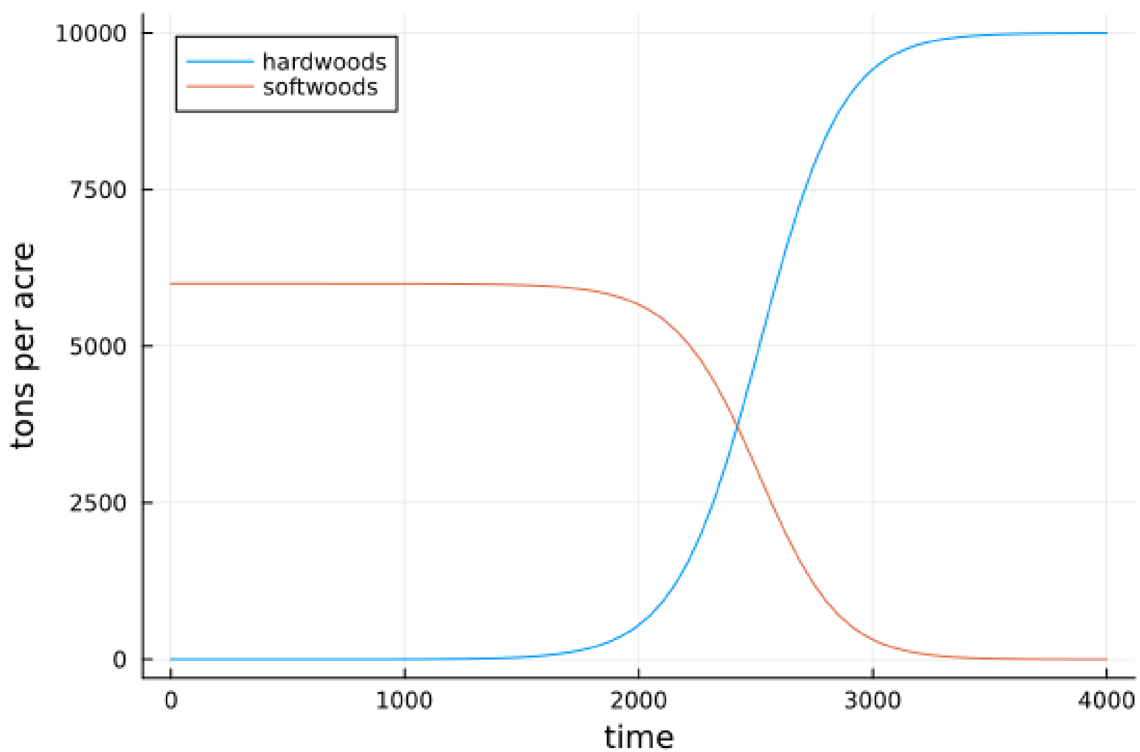
Out[18]:



In [19]:

```
plot(h*0:N,first.(Xs),label="hardwoods",
    xlabel="time",ylabel="tons per acre")
plot!(h*0:N,last.(Xs),label="softwoods")
```

Out[19]:



The number of hardwood trees starts small and grows slowly but after some time the the exponential growth takes over. Sometime later the increase in growth slows down and the forest reaches the new and stable equilibrium.

```
In [ ]:
```