

Math 420/620: Quiz 6 Solutions

**1.** The blue whale and fin whale are two similar species that inhabit the same areas. Hence, they are thought to compete. The intrinsic growth rate of each species is estimated at 5% per year for the blue whale and 8% per year for the fin whale. Suppose without competition that either species would undergo unlimited exponential growth. Thus, the growth rates for the populations  $B$  and  $F$  of blue and fin whales are

$$g_B(B) = 0.05B \quad \text{and} \quad g_F(F) = 0.08F.$$

Model the effects of competition as

$$c_B(B, F) = \alpha BF \quad \text{and} \quad c_F(B, F) = \alpha BF.$$

(i) Write the dynamical system governing the number of blue and fin whales as  $dX/dt = G(X)$  where  $X = (B, F)$ . What is  $G(X)$ ?

The rate of change in the population  $B$  of blue whales and  $F$  of fin whales is given by

$$\begin{aligned} dB/dt &= g_B(B) - c_B(B, F) = 0.05B - \alpha BF \\ dF/dt &= g_F(F) - c_F(B, F) = 0.08F - \alpha BF \end{aligned}$$

Now, let

$$X = \begin{bmatrix} B \\ F \end{bmatrix} \quad \text{and} \quad G(X) = G\left(\begin{bmatrix} B \\ F \end{bmatrix}\right) = \begin{bmatrix} 0.05B - \alpha BF \\ 0.08F - \alpha BF \end{bmatrix}.$$

Then the dynamical system governing the number of blue and fin whales can be written as  $dX/dt = G(X)$ .

(ii) Any point  $X$  such that  $G(X) = 0$  is an equilibrium state where the populations of blue and fin whales neither increase nor decrease over time. Find an equilibrium  $X_0$  such that  $B > 0$  and  $F > 0$ .

Setting  $G(X) = 0$  implies

$$0.05B - \alpha BF = 0 \quad \text{and} \quad 0.08F - \alpha BF = 0.$$

Since  $B > 0$  and  $F > 0$  divide the first equation by  $B$  and the second by  $F$  to obtain

$$0.05 - \alpha F = 0 \quad \text{and} \quad 0.08 - \alpha B = 0.$$

Therefore  $F_0 = 0.05/\alpha$  and  $B_0 = 0.08/\alpha$ .

It follows that

$$X_0 = \begin{bmatrix} B_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 0.08/\alpha \\ 0.05/\alpha \end{bmatrix}$$

is an equilibrium such that  $B > 0$  and  $F > 0$ .

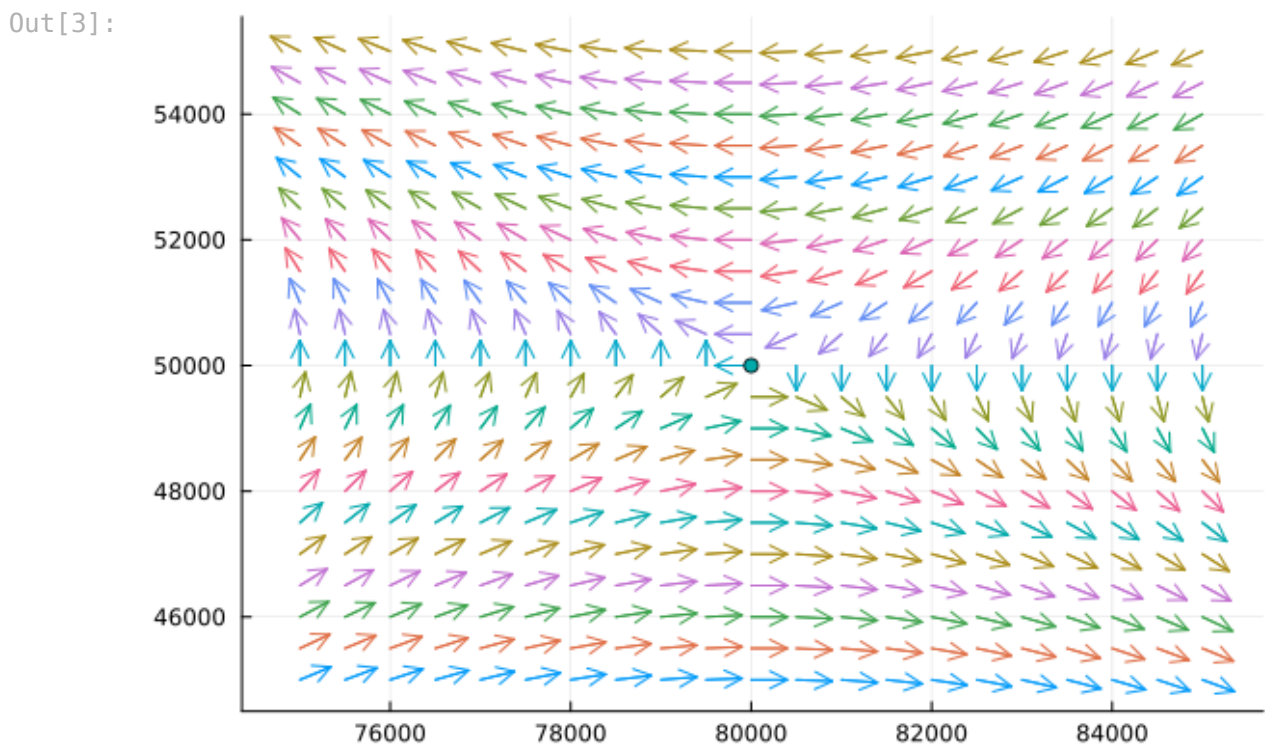
(iii) For  $\alpha = 10^{-6}$  plot the direction field of  $dX/dt = G(X)$  near  $X_0$ .

```
In [1]: alpha=1e-6
G(B,F)=[0.05*B-alpha*B*F,0.08*F-alpha*B*F]
X0=[0.08/alpha,0.05/alpha]
```

```
Out[1]: 2-element Vector{Float64}:
 80000.0
 50000.000000000001
```

```
In [2]: using Plots, LinearAlgebra
```

```
In [3]: Bs=X0[1]-5000:500:X0[1]+5000
Fs=X0[2]-5000:500:X0[2]+5000
quiver(
    Bs*ones(length(Fs))',
    ones(length(Bs))*Fs',
    quiver=(B,F)->G(B,F)/norm(G(B,F))*400
)
scatter!((X0[1],X0[2]),legend=false)
```



(iv) Compute the Jacobian matrix

$$DG(X_0) = \begin{bmatrix} \partial G_1 / \partial B & \partial G_1 / \partial F \\ \partial G_2 / \partial B & \partial G_2 / \partial F \end{bmatrix} \Big|_{(B,F)=X_0}$$

and find the eigenvalues of  $DG(X_0)$ . Determine whether the equilibrium state  $X_0$  is stable or unstable.

```
In [4]: using Symbolics
```

```
In [5]: @variables B,F
```

```
Out[5]: 2-element Vector{Num}:
         B
         F
```

```
In [6]: G(B,F)
```

```
Out[6]: 2-element Vector{Num}:
         0.05B - 1.0e-6B*F
         0.08F - 1.0e-6B*F
```

```
In [7]: DGnum=Symbolics.jacobian(G(B,F),[B,F])
         DGs="DG(B,F)="*string(DGnum)
         eval(Meta.parse(DGs))
```

```
Out[7]: DG (generic function with 1 method)
```

```
In [8]: Anum=DG(X0...)
         A=eval(Symbolics.toexpr(Anum))
```

```
Out[8]: 2×2 Matrix{Float64}:
         0.0  -0.08
        -0.05  0.0
```

```
In [9]: lambda=eigvals(A)
```

```
Out[9]: 2-element Vector{Float64}:
        -0.0632455532033676
         0.06324555320336758
```

The eigenvalues of  $DG(X_0)$  are

$$\lambda_1 \approx -0.06325 \quad \text{and} \quad \lambda_2 \approx 0.06325.$$

Since  $\lambda_2$  has positive real part then the equilibrium  $X_0$  is not stable.

(vi) Do the eigenvalues found above depend on  $\alpha$ . Give an intuitive explanation for why they should or why they should not.

```
In [10]: @variables alpha
```

```
Out[10]: 1-element Vector{Num}:
         alpha
```

```
In [11]: G(B,F)
```

```
Out[11]: 2-element Vector{Num}:  
 0.05B - B*F*alpha  
 0.08F - B*F*alpha
```

```
In [12]: DGnum=Symbolics.jacobian(G(B,F),[B,F])  
DGs="DG(B,F)="*string(DGnum)  
eval(Meta.parse(DGs))
```

```
Out[12]: DG (generic function with 1 method)
```

```
In [13]: DG(B,F)
```

```
Out[13]: 2x2 Matrix{Num}:  
 0.05 - F*alpha      -B*alpha  
 -F*alpha  0.08 - B*alpha
```

```
In [14]: X0=[0.08/alpha,0.05/alpha]
```

```
Out[14]: 2-element Vector{Num}:  
 0.08 / alpha  
 0.05 / alpha
```

```
In [15]: DG(X0...)
```

```
Out[15]: 2x2 Matrix{Num}:  
 0.0  -0.08  
 -0.05  0.0
```

Since

$$DG(B, F) = \begin{bmatrix} 0.05 - F\alpha & -B\alpha \\ -F\alpha & 0.08 - B\alpha \end{bmatrix},$$

then plugging  $X_0 = (0.08/\alpha, 0.05/\alpha)$  into the Jacobian matrix yields

$$DG(X_0) = \begin{bmatrix} 0.0 & -0.08 \\ -0.05 & 0.0 \end{bmatrix}.$$

Since  $DG(X_0)$  does not depend on  $\alpha$  then the eigenvalues do not depend on  $\alpha$ .

Note also that the stability of the fixed point  $X_0$  also does not depend on  $\alpha$ .

Consider the original dynamical system

$$dB/dt = 0.05B - \alpha BF$$

$$dF/dt = 0.08F - \alpha BF$$

and rescale the variables as  $\beta = \alpha B$  and  $\phi = \alpha F$ .

Therefore  $dB/dt = \alpha^{-1}d\beta/dt$  and  $dF/dt = \alpha^{-1}d\phi/dt$ .

It follows that

$$\alpha^{-1}d\beta/dt = 0.05(\alpha^{-1}\beta) - \alpha(\alpha^{-1}\beta)(\alpha^{-1}\phi)$$
$$\alpha^{-1}d\phi/dt = 0.08(\alpha^{-1}\phi) - \alpha(\alpha^{-1}\beta)(\alpha^{-1}\phi).$$

Now, multiply through by  $\alpha$  to obtain

$$d\beta/dt = 0.05\beta - \beta\phi$$
$$d\phi/dt = 0.08\phi - \beta\phi.$$

The above equation doesn't depend on  $\alpha$  but has the same dynamics as the original. This is another way to see why dynamical properties like stability at the fixed point don't depend on  $\alpha$ .

In [ ]: