Math 420/620: Quiz 7 Solutions

1. Astronauts in training are required to practice a docking maneuver under manual control. As a part of this maneuver, it is required to bring an orbiting spacecraft to rest relative to another orbiting craft. The hand controls provide for variable acceleration and deceleration, and there is a device on board that measures the rate of closing between the two vehicles.

Suppose at time t_0 the initial closing velocity of the spacecraft is v_0 and the thruster rocket is providing an acceleration of a_0. The following strategy has been proposed for bringing the craft to rest. At time t_n observe the closing velocity v_n. If it is zero, we are done. Otherwise, prepare to set the acceleration control. After a delay of c seconds set the thruster to $a_{n+1} = -\kappa v_n$. Wait another w seconds until time t_{n+1} and then repeat the procedure.

Since $dv/dt = a$ it follows that

$$v_{n+1} = \begin{cases} v_0 + ca_0 - w\kappa v_0 & \text{for } n = 0 \\ v_n - c\kappa v_{n-1} - w\kappa v_n & \text{for } n > 0. \end{cases}$$

Upon setting $X_n = (v_n, v_{n-1})$ we obtain $X_{n+1} = AX_n$ where

$$A = \begin{bmatrix} 1 - w\kappa & -c\kappa \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad X_1 = \begin{bmatrix} v_0 + ca_0 - w\kappa v_0 \\ v_0 \end{bmatrix}.$$

(i) Let $c = 5$, $w = 10$ and $\kappa = 0.07$. Draw the vector field of $F(X) = (A - I)X$ for $X \in \{-3, -2, \ldots, 3\}^2$ as in Figure 4.9 from the text.

In [1]:
```
c=5
w=10
kappa=0.07
A=[1-w*kappa -c*kappa; 1 0]
```

Out[1]:
```
2×2 Matrix{Float64}:
 0.3  -0.35
 1.0   0.0
```

In [2]:
```
using LinearAlgebra
```

In [3]:
```
F(X)=(A-I)*X
```

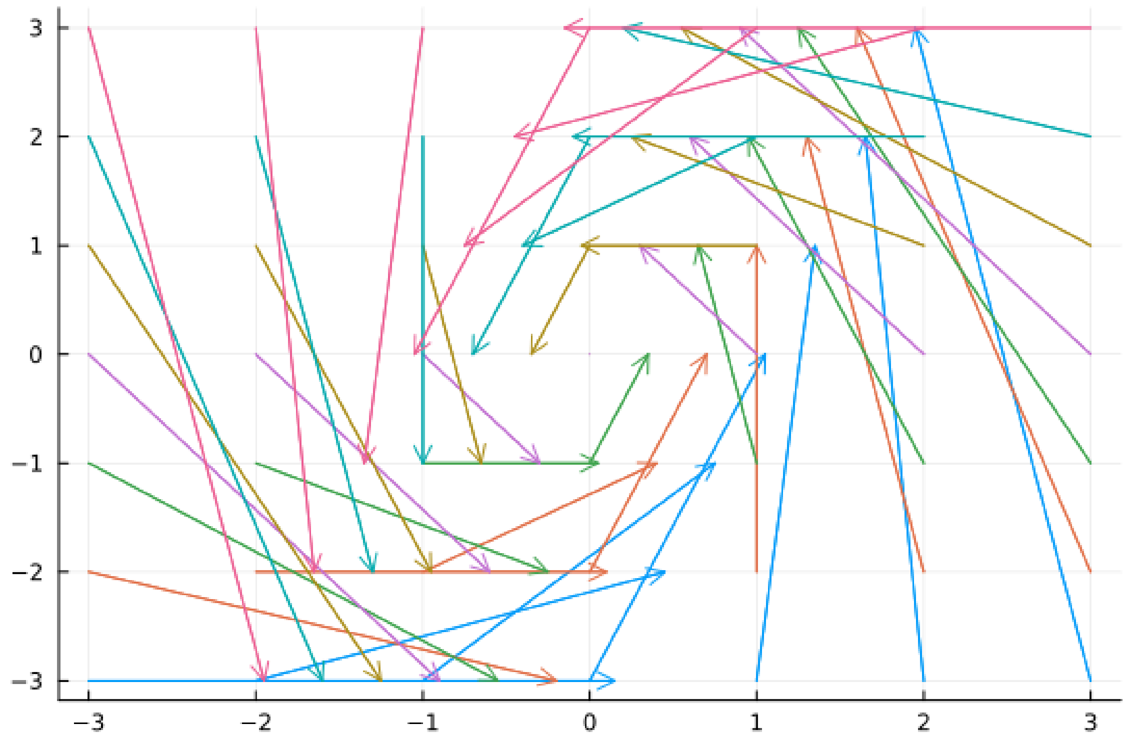
Out[3]:
```
F (generic function with 1 method)
```

In [4]:
```
using Plots
```

In [5]:
```
xs=-3:1:3; ys=-3:1:3
quiver(
```

```
    xs*ones(length(ys))',ones(length(xs))*ys',
    quiver=(x,y)->F([x,y])
)
```

Out[5]:



(ii) Consider the continuous-time analogue $dX/dt = F(X)$ of the docking problem. Find the eigenvalues $\mu_1 = \alpha_1 + i\beta_1$ and $\mu_2 = \alpha_2 + i\beta_2$ of $A - I$. Recall if the real parts $\alpha_1 < 0$ and $\alpha_2 < 0$, then $X(t) \to 0$ as $t \to \infty$. Does $X(t) \to 0$?

In [6]:

```
mu=eigvals(A-I)
```

Out[6]:

```
2-element Vector{ComplexF64}:
 -0.8500000000000001 - 0.57227615711298im
 -0.8500000000000001 + 0.57227615711298im
```

The real parts of the eigenvalues are $\alpha_1 = -0.85$ and $\alpha_2 = -0.85$. This implies that $X(t) \to 0$ and $t \to \infty$.

(iii) Vary κ while fixing $c = 5$ and $w = 10$. Let $\mu_1(\kappa)$ and $\mu_2(\kappa)$ be the corresponding eigenvalues of $A - I$. Plot the real parts $\alpha_1(\kappa)$ and $\alpha_2(\kappa)$ for $\kappa \in [0, 0.5]$.

Let $B(\kappa) = A - I$ for the value of κ given with the other arguments fixed.

In [7]:

```
B(kappa)=[1-w*kappa -c*kappa; 1 0]-I
```

Out[7]:

```
B (generic function with 1 method)
```

In [8]:
```julia
# Check that the B function works
B(0.05)
```

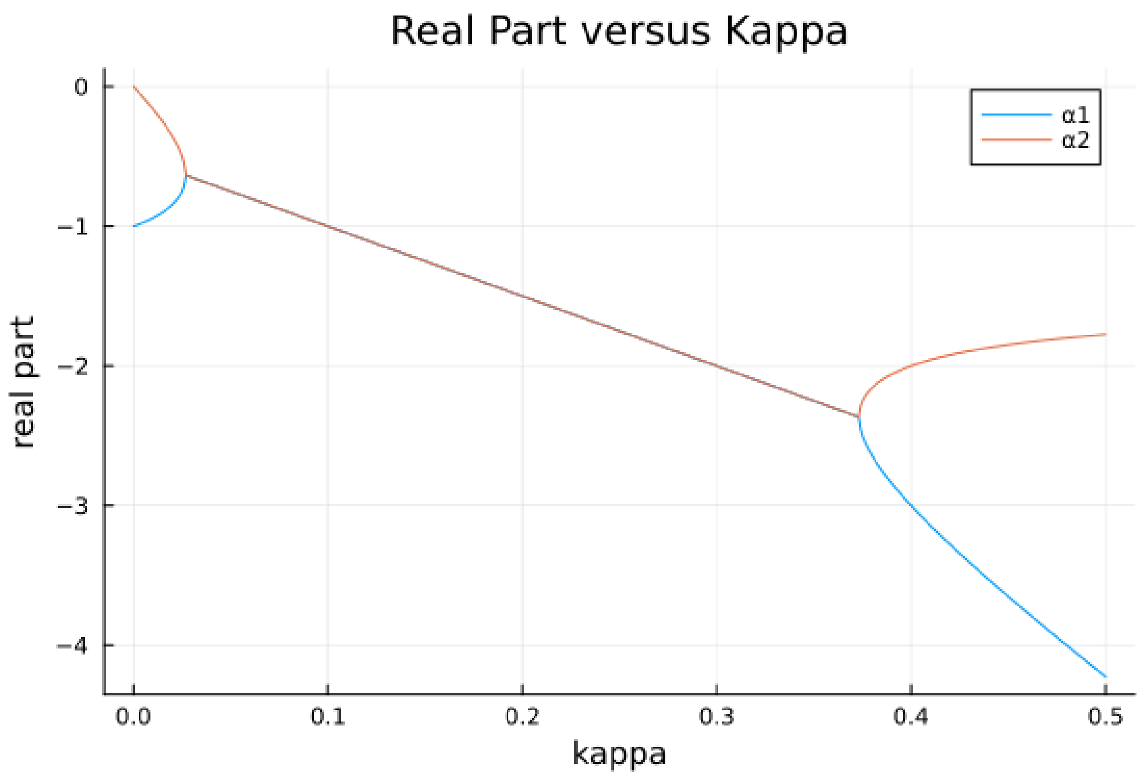
Out[8]:
```
2×2 Matrix{Float64}:
 -0.5  -0.25
  1.0  -1.0
```

In [9]:
```julia
alpha1(kappa)=real(eigvals(B(kappa))[1])
alpha2(kappa)=real(eigvals(B(kappa))[2])
```

Out[9]:
```
alpha2 (generic function with 1 method)
```

In [10]:
```julia
plot([alpha1,alpha2],0:0.001:0.5,
    xlabel="kappa",ylabel="real part",
    title="Real Part versus Kappa",
    label=["α1" "α2"])
```

Out[10]:



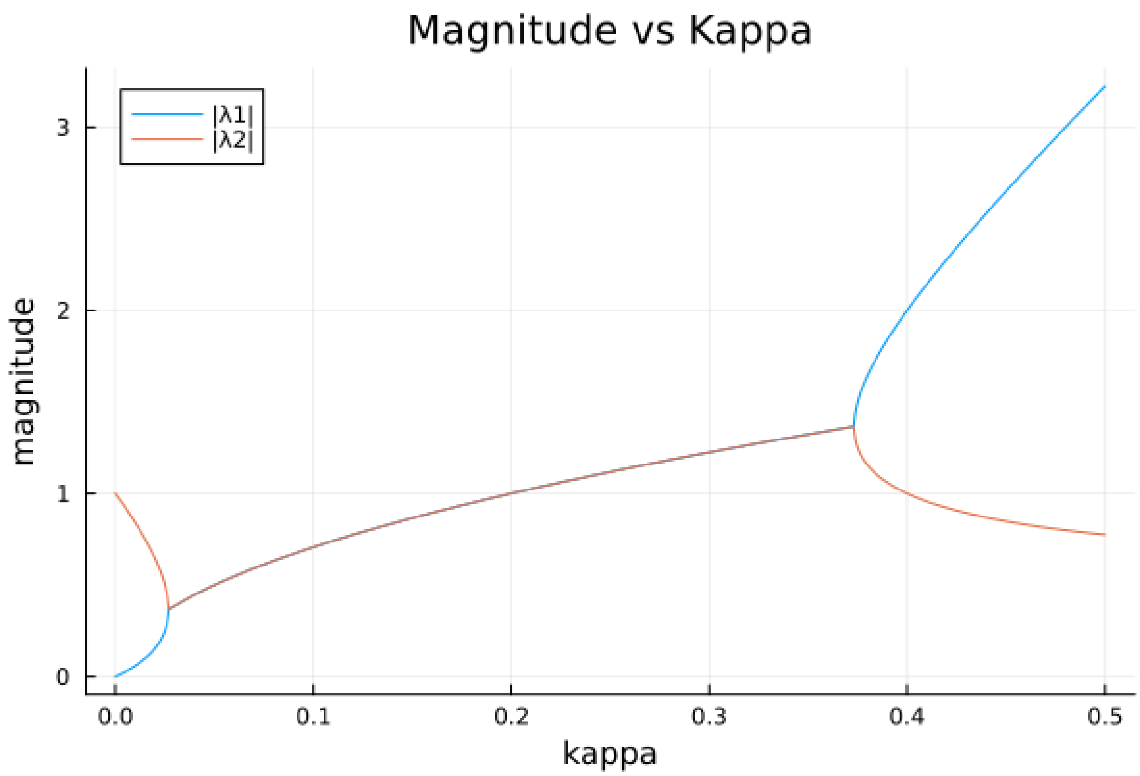
(iv) For what values of κ does $X(t) \to 0$? Compare your result with the eigenvalue analysis in Lecture 22 for the discrete system $X_{n+1} = AX_n$. Comment on the differences between the continuous and discrete models.

As seen by the graph $\alpha_1 < 0$ and $\alpha_2 < 0$ for all values of $\kappa \in (0, 0.5]$. Therefore, for the continuous system $X(t) \to 0$ as $t \to \infty$ for all values of $\kappa > 0$.

Recall for the the discrete model $X_{n+1} = AX_n$ that $X_n \to 0$ provided both $|\lambda_1| < 1$ and $|\lambda_2| < 1$ where λ_1 and λ_2 are the eigenvalues of A. The corresponding plot for the discrete system is given as follows:

In [11]:
```
fA(kappa)=[1-w*kappa -c*kappa; 1 0]
fabseig1(kappa)=abs(eigvals(fA(kappa))[1])
fabseig2(kappa)=abs(eigvals(fA(kappa))[2])
plot([fabseig1,fabseig2],0:0.001:0.5,
    xlabel="kappa",ylabel="magnitude",
    title="Magnitude vs Kappa",
    label=["|λ1|" "|λ2|"])
```

Out[11]:



As seen by the plot $|\lambda_1| < 1$ and $|\lambda_2| < 1$ for $\kappa \in (0, 0.2)$. Therefore, for the discrete system $X_n \to 0$ as $n \to \infty$ only when $\kappa \in (0, 0.2)$.

The difference between the continuous and the discrete models is that the continous docking model succeeds for more values of κ. Since the continuous model is a further simplification of the discrete model, we conclude that important properties have been altered by this simplication. In particular, the continuous model implying that the docking model succeeds in casses where the discrete model does not imply it succeeds could lead to wrong decisions on how to set the acceleration control κ of the spacecraft.

In [ ]: