Math 420/620 Quiz 9

1. This problem studies the striking difference between the behavior of continuous-time and discrete-time dynamical systems that can occur even in simple models.

(i) Show that the continuous-time dynamical system

$$\frac{dy}{dt} = (a-1)y - ay^2$$

has an equilibrium at $y = (a-1)/a$ for any $a > 1$.

We let $g(y) = (a-1)y - ay^2$ and solve $g(y) = 0$.

Since

$$g(y) = (a-1)y - ay^2 = y(a-1-ay) = 0$$

then either $y = 0$ or $a - 1 - ay = 0$.

The second possibility is satisfied when $y = (a-1)/a$. Therefore that is a fixed point.

Note there is another fixed point at $y = 0$.

(ii) Linearize the dynamical system in part (i) about the equilibrium and show the fixed point is stable for all $a > 1$.

Let $p = (1-a)/a$ be the fixed point. The linearized right hand size is given by

$$g_L(y) = g(p) + g'(p)(y-p).$$

Since $g'(y) = a - 1 - 2ay$, then $a > 1$ implies

$$g'(p) = a - 1 - 2ap = a - 1 - 2(a-1) = 1 - a < 0.$$

It follows that the fixed point is linearly stable and consequently stable.

(iii) Show that the analogous discrete-time dynamical system

$$\Delta z_n = (a-1)z_n - az_n^2$$

also has an equilibrium at $z = (a-1)/a$ for any $a > 1$.

Note that $\Delta z_n = g(z_n)$ where g is defined as in (i). Therefore, the points such that $g(z) = 0$ are also the same. Thus $z = (1-a)/a$ as well as $z = 0$ are equilibria points.

(iv) Write the discrete-time dynamical system in (iii) as $z_{n+1} = f(z_n)$. What is $f(z)$?

Since $\Delta z_n = z_{n+1} - z_n$ it follows that

$$z_{n+1} = z_n + \Delta z_n = z_n + (a-1)z_n - az_n^2 = az_n - az_n^2 = f(z_n)$$

where $f(z) = az(1-z)$.

(v) The fixed point $z = (a-1)/a$ is stable when $|f'(z)| < 1$ at that fixed point. For what values of $a > 1$ is the fixed point stable.

Differentiating yields

$$f'(z) = \frac{d}{dz}\left(az - az^2\right) = a - 2az.$$

Again let $p = (a-1)/a$. Then

$$f'(p) = a - 2ap = a - 2(a-1) = -a + 2.$$

Solving $|f'(p)| = |a-2| < 1$ implies

$$-1 < a - 2 < 1 \quad \text{or} \quad 1 < a < 3.$$

Consequently the fixed point $z = (a-1)/a$ is stable when $a \in (1, 3)$.

(vi) Let $a = 2.8$ and $z_0 = 0.7$. Make a scatter plot of the points (n, z_n) for $n = 0, 1, \ldots, 30$. Describe what happened.

In [1]:
```
using Plots
```

In [2]:
```
function mkpoints(a,z0)
    f(z)=a*z*(1-z)
    zs=[z0]
    zn=z0
    for n=1:30
        zn=f(zn)
        push!(zs,zn)
    end
    return zs
end
```

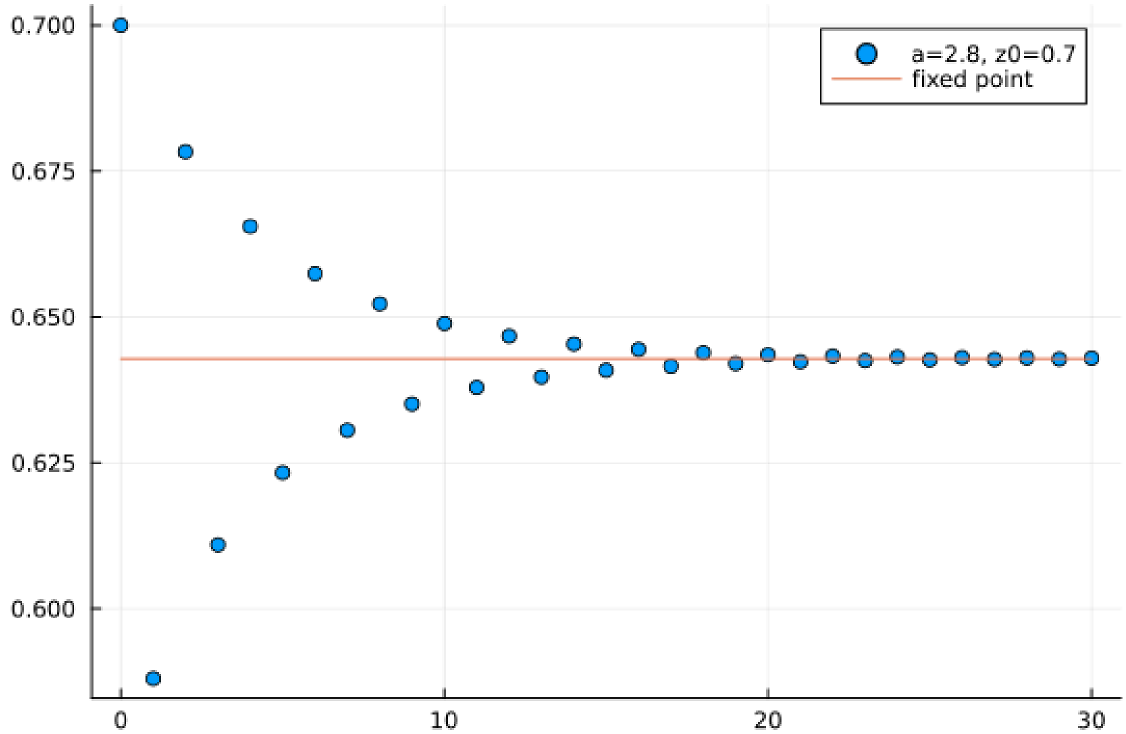
Out[2]:
```
mkpoints (generic function with 1 method)
```

In [3]:
```
a=2.8; p=(a-1)/a
```

Out[3]:
```
0.6428571428571428
```

In [4]:
```
scatter(0:30,mkpoints(a,0.7),label="a=2.8, z0=0.7")
plot!([0,30],[p,p],label="fixed point")
```

Out[4]:



The discrete dynamical system converged to the fixed point at

$$p = (a-1)/a \approx 0.6428571428571428.$$

This makes sense because $a = 2.8$ implies $a \in (1, 3)$ so the fixed point is stable.

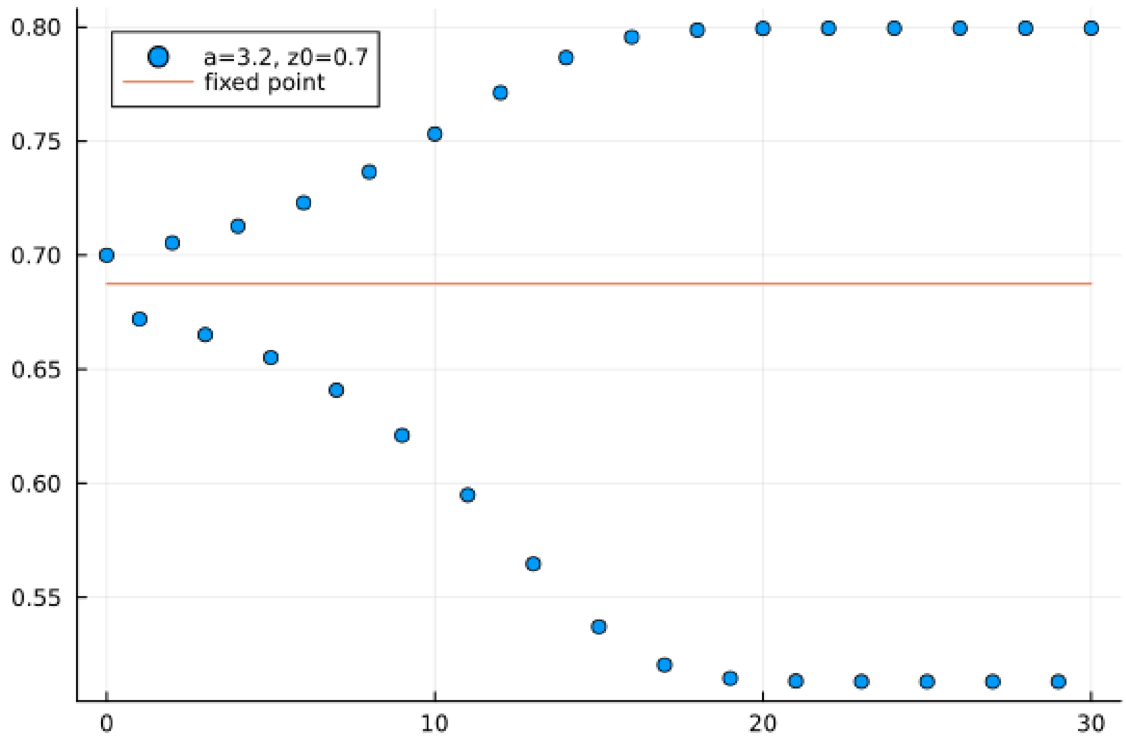
(vii) Let $a = 3.2$ and $z_0 = 0.7$. Make a scatter plot of the points (n, z_n) for $n = 0, 1, \ldots, 30$. Describe what happened.

In [5]:
```
a=3.2; p=(a-1)/a
```

Out[5]:
```
0.6875
```

In [6]:
```
scatter(0:30,mkpoints(a,0.7),label="a=3.2, z0=0.7")
plot!([0,30],[p,p],label="fixed point")
```

Out[6]:



When $a = 3.2$ then $a \neq (1, 3)$ and the fixed point is unstable. In this case the dynamical system does not converge to the fixed point. However, as it converges it appears to oscillate between two points $p_1 \approx 0.8$ and $p_2 \approx 0.51$.

This is a cycle a length two.

The following is an optional discussion about and description of the limit cycle. I decided to more accurately compute the points p_1 and p_2 that the limit converges two. The values of p_1 and p_2 can be obtained by running the dynamical system for a long time and examining the points z_n and z_{n+1} for large n. Instead of that, I used Newton's method to solve $f(f(z)) = z$.

A cycle of length two satisfies the equation $f(f(z)) = z$. Substituting yields

$$f(f(z)) = f(az(1-z)) = a(az(1-z))(1 - az(1-z)) = z.$$

Therefore

$$\begin{aligned} a^2 z(1-z)(1-az(1-z)) &= a^2 z(1 - z - az(1-z)^2) \\ &= a^2 z(1 - z - az(z^2 - 2z + 1)) \\ &= a^2 z(1 - (1+a)z + 2az^2 - az^3) = z. \end{aligned}$$

Assuming $z \neq 0$ then

$$1 - (1+a)z + 2az^2 - az^3 = 1/a^2.$$

Consequently

$$h(z) = 0 \quad \text{where} \quad h(z) = 1 - 1/a^2 - (1+a)z + 2az^2 - az^3.$$

Now look for roots p_1 and p_1 of h near 0.8 and 0.51 respectively using Newton's method

$$w_{k+1} = w_k - h(w_k)/h'(w_k).$$

Note that differentiating yields

$$h'(z) = -1 - a + 4az - 3az^2.$$

In [7]:
```
h(z)=1-1/a^2-z*(1+a-z*(2*a-a*z))
dh(z)=-1-a+z*(4*a-3*a*z)
```

Out[7]:
```
dh (generic function with 1 method)
```

In [8]:
```
wk=0.8
for k=1:5
    wk=wk-h(wk)/dh(wk)
    println("w$k=",wk)
end
p1=wk
```

```
w1=0.7994591346153838
w2=0.7994554906323398
w3=0.7994554904673702
w4=0.7994554904673702
w5=0.7994554904673702
```

Out[8]:
```
0.7994554904673702
```

In [9]:
```
wk=0.51
for k=1:5
    wk=wk-h(wk)/dh(wk)
    println("w$k=",wk)
end
p2=wk
```

```
w1=0.5129625355113645
w2=0.5130444476239788
w3=0.5130445095325955
w4=0.5130445095326303
w5=0.5130445095326303
```

Out[9]:
```
0.5130445095326303
```

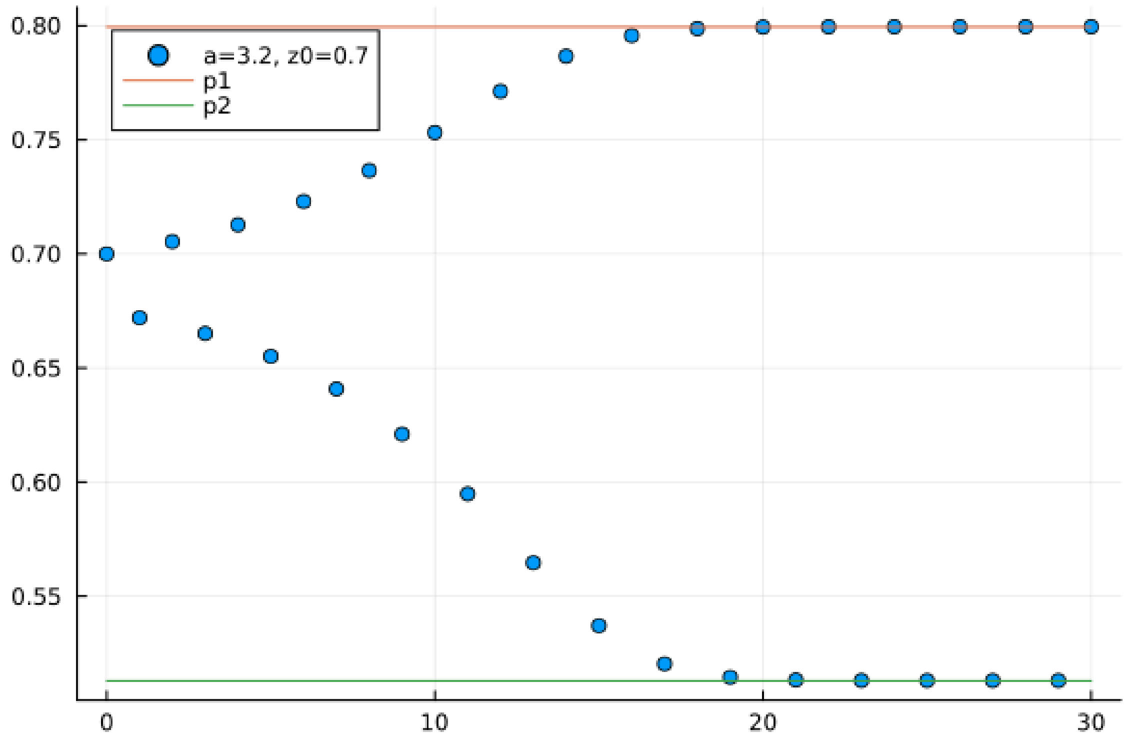
It follows that when $a = 3.2$ the dynamical systems converges to a cycle between the two points

$$p_1 \approx 0.7994554904673702 \quad \text{and} \quad p_2 \approx 0.5130445095326303.$$

In [10]:
```
scatter(0:30,mkpoints(a,0.7),label="a=3.2, z0=0.7")
plot!([0,30],[p1,p1],label="p1")
```

```
plot!([0,30],[p2,p2],label="p2")
```

Out[10]:



(viii) Let $a = 4.0$ and $z_0 = 0.7$. Make a scatter plot of the points (n, z_n) for $n = 0, 1, \ldots, 30$. Describe what happened.

In [11]:

```
a=4.0; p=(a-1)/a
```

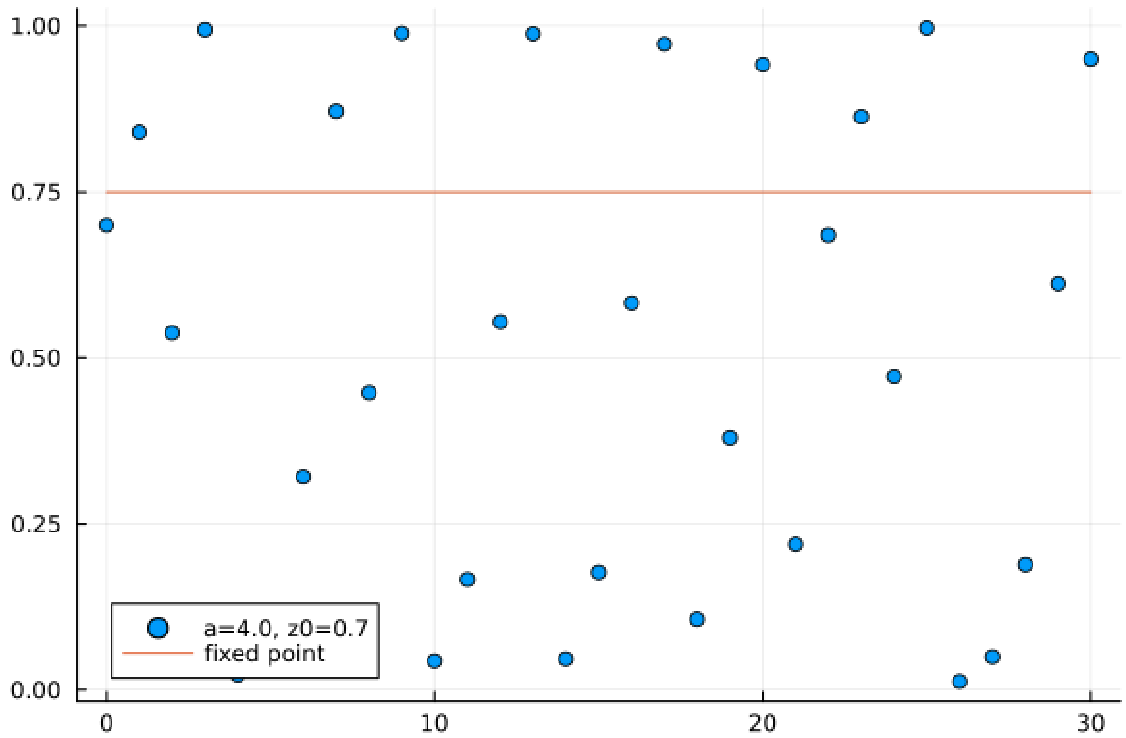
Out[11]:

```
0.75
```

In [12]:

```
scatter(0:30,mkpoints(a,0.7),label="a=4.0, z0=0.7")
plot!([0,30],[p,p],label="fixed point")
```

Out[12]:



When $a = 4$ the distribution of the points z_n appear random on the interval $(0, 1)$. This looks like chaos to me.

In [ ]: