Math 466/666: Programming Project 2

This project explores the use of Hermite interpolation in the context solving for the roots of $f(x) = 0$ and the difficulties which happen with a repeated root.

For this project the class has not been randomly grouped into teams, instead, please form your own teams consisting of 3 or 4 students. Each team should present their work in the form of a typed report using clear and properly punctuated English. Pencil and paper calculations may be typed or hand written. Where appropriate include full program listings and output.

Every team member should participate in the work and be prepared to independently answer questions concerning the material. One report per team must be submitted through WebCampus to complete this project. Make sure the report addresses each of the items listed below and then upload the report as a single pdf file.

1. List the members on your team and explain how the work for this project was conducted. Please provide details concerning how many meetings were held, what was discussed at each meeting and what work was done between meetings.

Further include a written statement attesting that the submitted report represents the original efforts of the team members listed and, in particular, does contain the work of other students in the class.

- **(i)** It is fine to consult books, published papers and online resources while working on this project. If you do so, every outside source of information should be cited with a proper bibliographic reference at the point it is used.
- **(ii)** There is no need to cite any additional help provided by the instructor during class or individually during office hours.

2. Let x_1 and x_2 be distinct. The corresponding Lagrange basis functions are given by

$$\ell_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad \text{and} \qquad \ell_2(x) = \frac{x - x_1}{x_2 - x_1}.$$

Consider the following polynomials $\eta_1(x) = \ell_1(x)^2 \ell_2(x)$ and $\eta_2(x) = \ell_1(x)\ell_2(x)^2$. Use calculus to find $\eta_i(x_j)$ and $\eta_i'(x_j)$ for $i = 1, 2$ and $j = 1, 2$.

3. Let A and B be constants and define $g_1(x) = A\eta_1(x)$ and $g_2(x) = B\eta_2(x)$. Solve for A and B such that

$$g_1(x_1) = 0, \quad g_1(x_2) = 0, \quad g_1'(x_1) = 1 \quad \text{and} \quad g_1'(x_2) = 0$$

and

$$g_2(x_1) = 0, \quad g_2(x_2) = 0, \quad g_2'(x_1) = 0 \quad \text{and} \quad g_2'(x_2) = 1.$$

4. Let a and b be constants and consider the polynomials

$$h_1(x) = \ell_1(x) + a\big(g_1(x) + g_2(x)\big) \quad \text{and} \quad h_2(x) = \ell_2(x) + b\big(g_1(x) + g_2(x)\big).$$

Solve for a and b such that

$$h_1(x_1) = 1, \quad h_1(x_2) = 0, \quad h_1'(x_1) = 0 \quad \text{and} \quad h_1'(x_2) = 0$$

and

$$h_2(x_1) = 0, \quad h_2(x_2) = 1, \quad h_2'(x_1) = 0 \quad \text{and} \quad h_2'(x_2) = 0.$$

5. Let y_i and z_i for $i = 1, 2$ be constants. Show that the polynomial given by

$$p(x) = \sum_{i=1}^{2} \Big\{ y_i h_i(x) + z_i g_i(x) \Big\}$$

satisfies the conditions $p(x_i) = y_i$ and $p'(x_i) = z_i$ for $i = 1, 2$ and explain why $p(x)$ is the unique polynomial of minimal degree that satisfies those conditions.

6. Let $f(x) = e^x - 13/x$ and define $y_i = f(x_i)$ and $z_i = f'(x_i)$ for $i = 1, 2$ where $x_1 = 1.3$ and $x_2 = 2.7$. Write a computer program that computes $p(x)$ for any value of x, verify $p(2) \approx 0.9393631693832503$ and then compute the size of the error $e = |f(2) - p(2)|$.

7. Plot $f(x)$ and $p(x)$ on the same graph over the interval $[0.5, 4]$. You might find the plotting commands

```
xs=[0.5:0.1:4;];
yf=f.(xs);
yp=p.(xs);
plot(xs,[yf yp],size=[400,300],
    legend=:bottomright,label=["f" "p"])
```

to be useful when making the graph in Julia.

8. [Extra Credit] Modify the program in Question 6 to estimate the maximum error

$$E = \max \big\{ \, |f(x) - p(x)| : x \in [1, 3] \, \big\}$$

good to 4 significant digits. Use any reasonable method you can find to compute E and explain how it works.

9. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuously differentiable such that $f'(x) \neq 0$ for $x \in \mathbf{R}$. By the inverse function theorem f is invertible and $(f^{-1})'(y) = 1/f'(f^{-1}(y))$. We may use this formula to adapt the polynomial developed above to interpolate the inverse function f^{-1} by swapping the roles of x_i, y_i and z_i as

original	replacement
x_i	y_i
y_i	x_i
z_i	$1/z_i$

Thus, everywhere there used to be x_i write y_i, everywhere there used to be y_i write x_i and replace z_i by $1/z_i$. Let $q(y)$ be the polynomial obtained in this way which interpolates $f^{-1}(y)$ with $x_1 = 1.3$ and $x_2 = 2.7$ where $f(x)$ is the function from Question 6. Verify that $q(0) \approx 1.8849155134426776$.

10. Consider the two-step Newton-like method for approximating the root of $f(x) = 0$ obtained by iterating the inverse interpolation in Question 9 as follows:

Inverse Hermite Method. Let $q_n(y)$ be the polynomial such that

$$q_n(y_i) = x_i \quad \text{and} \quad q_n'(y_i) = 1/z_i \qquad \text{for} \qquad i = n, n+1.$$

Define

$$x_{n+2} = q_n(0), \quad y_{n+2} = f(x_{n+2}) \quad \text{and} \quad z_{n+2} = f'(x_{n+2}).$$

Now repeat until x_n converges.

Note that $x_3 \approx 1.8849155134426776$ has already been computed above. Write a program to perform five iterations of the inverse Hermite method for the function $f(x)$ given in Question 6 starting with $x_1 = 1.3$ and $x_2 = 2.7$. Report the values of $x_3, \ldots, x_7$. Did the method converge? If so, to what did it converge?

11. [Extra Credit] Theoretically, what should the order of convergence be for the inverse Hermite method? Use big-number arithmetic to verify your prediction.

12. By definition, if α is a root of higher multiplicity, then $f(\alpha) = 0$ and also $f'(\alpha) = 0$. Let $f(x) = x^4 - 4x^2 + 4$ and use calculus to show this function has a root $\alpha \in [1, 2]$ of higher multiplicity. What is the exact value of α?

13. When $f(x)$ has a root α of higher multiplicity, then it may not be invertible near $x = \alpha$ and consequently approximating the inverse function may not make sense. Moreover, since $z_n \to 0$ as $x_n \to \alpha$ there could be numerical difficulties when calculating the quantities $1/z_n$. Let $f(x)$ be as in Question 12 and perform twenty steps of the inverse Hermite method starting with $x_1 = 1$ and $x_2 = 2$ to find $x_3, \ldots, x_{22}$.

14. Compute the errors $e_n = |x_n - \alpha|$ for $n = 3, \ldots 22$ where α is the root found earlier. Does the method converge? If so, what is the order of convergence?

15. [Extra Credit] When $f(x)$ has a root of higher multiplicity, the simple trick of setting $r(x) = f(x)/f'(x)$ and then searching for the roots of $r(x)$ often leads to fast convergence. Use the inverse Hermite method to solve $r(x) = 0$ where $f(x)$ is the same function as in Question 12. Did you get the same root? Was the convergence faster? Can you think of anything that might make this idea work better?