Math 466/666: Quiz 1 Version A

This is a closed-book closed-notes quiz monitored through Zoom. Please enable both your web camera and your screen share during the quiz. You may send the instructor a private message to the instructor if you have a question or find an error in the quiz; otherwise, do not send messages in chat. Work each problem using pencil and paper on a clean sheet of paper. Be sure to write your name on each sheet of paper!

When you are finished use the raise hand feature of Zoom and I will move you to a breakout room where you can show me your student ID and completed work. Do not leave Zoom without first showing me your work in the breakout room. After you are done in the breakout room, please log out of Zoom and upload a high-resolution version of your work for grading to WebCampus. It is extremely important that you not make any changes in your answers before uploading them to WebCampus.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. It is known that the matrix 1-norm and ∞-norms are given by the formulas

$$\|A\|_1 = \max\Big\{ \sum_{i=1}^{n} |A_{ik}| : k = 1,\ldots,n \Big\}$$

and

$$\|A\|_\infty = \max\Big\{ \sum_{j=1}^{n} |A_{kj}| : k = 1,\ldots,n \Big\}.$$

In other words, $\|A\|_1$ is the maximum of sums of absolute values taken along columns and $\|A\|_\infty$ is the maximum of sums of absolute values taken along rows. Suppose

$$A = \begin{bmatrix} 5 & -4 & 7 \\ 1 & -8 & -6 \\ -5 & 4 & 6 \end{bmatrix}.$$

(i) Show all the work needed to find $\|A\|_1$.

(ii) Show all the work needed to find $\|A\|_\infty$.

3. Given $A \in \mathbf{R}^{n\times n}$ the induced or natural matrix p-norm $\|A\|_p$ is given as which one of the following:

(A) $\|A\|_p = \max\{\, \|x\|_p : \|Ax\|_p = 1\,\}$.

(B) $\|A\|_p = \min\{\, \|x\|_p : \|Ax\|_p = 1\,\}$.

(C) $\|A\|_p = \max\{\, \|Ax\|_p : \|x\|_p = 1\,\}$.

(D) $\|A\|_p = \min\{\, \|Ax\|_p : \|x\|_p = 1\,\}$.

(E) none of the above.

As your answer, write down down the entire text for your choice including the letter and the definition on your paper. If you choose (E), also include a correct definition of $\|A\|_p$.

4. Newton's binomial theorem is the same as Taylor series for $(1+x)^\alpha$ expanded about $x = 0$. This is which one of the following:

(A) If $|x| > 1$ then $(1+x)^\alpha = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

(B) If $|x| < 1$ then $(1+x)^\alpha = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

(C) If $|x| > 1$ then $(1+x)^\alpha = \displaystyle\sum_{k=0}^{\infty} \binom{k}{\alpha} x^k$.

(D) If $|x| < 1$ then $(1+x)^\alpha = \displaystyle\sum_{k=0}^{\infty} \binom{k}{\alpha} x^k$.

(E) none of the above.

As your answer, write down down the entire text for your choice including the letter and statement on your paper. If you choose (E), also include a correct statement of Newton's binomial theorem.

5. Let $x_j = x_0 + jh$ where $h > 0$ and $f_j = f(x_j)$. Let $\Delta f_j = f_{j+1} - f_j$ be the forward difference operator. If $f(x)$ is a polynomial of degree n, explain why $\Delta^n f_j$ is a constant.

6. The interpolating polynomial $p(t)$ of degree $n-1$ passing through the points (x_j, y_j) for $j = 1, \ldots, n$ can be written as

$$p(t) = \sum_{j=1}^{n} y_j \ell_j(t)$$

where $\ell_j(t)$ are the Lagrange polynomial basis functions. What is the formula that determines $\ell_j(t)$?

7. The following theorem is missing details.

Theorem on Interpolating Polynomials: *Given the distinct points x_i where $i = 1, \ldots, n$, let $p(x)$ be the unique interpolating polynomial of degree less than or equal $n-1$ such that*

$$p(x_i) = f(x_i) \qquad \textit{for} \qquad i = 1, \ldots, n.$$

Provided f has n derivatives, then for every t there is a corresponding

ξ between $\boxed{\qquad\qquad}$ *and* $\boxed{\qquad\qquad}$ *such that*

$$f(t) = p(t) + \boxed{\qquad\qquad} \qquad \textit{where} \qquad q(t) = \prod_{i=1}^{n}(t - x_i).$$

As your answer, write down the entire theorem on your paper filling in the missing details. Your answer should start with "Theorem on Interpolating Polynomials" and end with the definition of $q(t)$.

1. I have worked independently on this quiz.
— Test Student.

2(i) The maximum of sums of absolute values taken along columns for

$$A = \begin{bmatrix} 5 & -4 & 7 \\ 1 & -8 & -6 \\ -5 & 4 & 6 \end{bmatrix}$$

is maximum

$7+6+6 = 19$

$4+8+4 = 16$

$5+1+5 = 11$

Thus $\|A\|_1 = 19$.

(ii) The maximum of sums of absolute values taken along rows for

$$A = \begin{bmatrix} 5 & -4 & 7 \\ 1 & -8 & -6 \\ -5 & 4 & 6 \end{bmatrix} \begin{matrix} \to 5+4+7 = 16 \\ \to 1+8+6 = 15 \\ \to 5+4+6 = 15 \end{matrix}$$

is maximum.

Thus $\|A\|_\infty = 16$.

3. The answer is

(C) $\|A\|_p = \max\{\|Ax\|_p : \|x\|_p = 1\}$.

4. The answer is

(B) If $|x|<1$, then $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

5. To see why $\Delta^n f_j$ is constant when f is a polynomial of degree n it is sufficient to look at the monomial term x^p and check that Δx_j^p is a polynomial of degree $p-1$. That this is sufficient then follows by induction on n. Now compute as

$$\Delta x_j^p = x_{j+1}^p - x_j^p = (x_j+h)^p - x_j^p$$

$$= \sum_{k=0}^{p} \binom{p}{k} x_j^{p-k} h^k - x_j^p$$

$$= \cancel{x_j^p} + \sum_{k=1}^{p} \binom{p}{k} x_j^{p-k} h^k - \cancel{x_j^p}$$

$$= \sum_{l=0}^{p-1} \binom{p}{l+1} x_j^{p-1-l} h^{l+1}$$

which is a polynomial of degree $p-1$ in x_j. To finish the explanation, note that any polynomial $f(x)$ of degree n is a sum of monomial terms

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

So that

$$\Delta f(x_j) = \Delta\left(a_n x_j^n + a_{n-1} x_j^{n-1} + \cdots + a_1 x_j + a_0\right)$$

$$= a_n \Delta x_j^n + a_{n-1} \Delta x_j^{n-1} + \cdots + a_1 \Delta x_j + \Delta a_0$$

Since $\Delta a_0 = 0$ and each of the other terms a polynomials no greater than degree $n-1$, then

$$\Delta f(x_j) = \text{polynomial in } x_j \text{ of degree } n-1$$

By induction it follows that

$$\Delta^2 f(x_j) = \text{polynomial in } x_j \text{ of degree } n-2$$

$$\vdots$$

$$\Delta^n f(x_j) = \text{polynomial of degree } n-n=0$$

which is a constant.

6. The Lagrange polynomial basis functions are

$$l_j(t) = \prod_{k \neq j} \frac{(t - x_k)}{(x_j - x_k)}.$$

7. Theorem on Interpolating Polynomials: Given the distinct points x_i where $i=1,\dots,n$, let $p(x)$ be the unique interpolating polynomial of degree less than or equal $n-1$ such that

$$p(x_i) = f(x_i) \quad \text{for} \quad i=1,\dots,n.$$

Provided f has n derivatives, then for every t there is a corresponding ξ between

$\min(t, x_1, x_2, \dots, x_n)$ and $\max(t, x_1, x_2, \dots, x_n)$

such that

$$f(t) = p(t) + \frac{q(t)}{n!} f^{(n)}(\xi) \quad \text{where } q(t) = \prod_{i=1}^{n} (t - x_i).$$