Newton's Method:

Let x_0 be an initial approximation of the solution to $f(x)=0$. Then define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

don't want this zero.

Q1: Do the x_n's converge to the solution?

Q2: How fast?

Q3: Under what hypothesis?

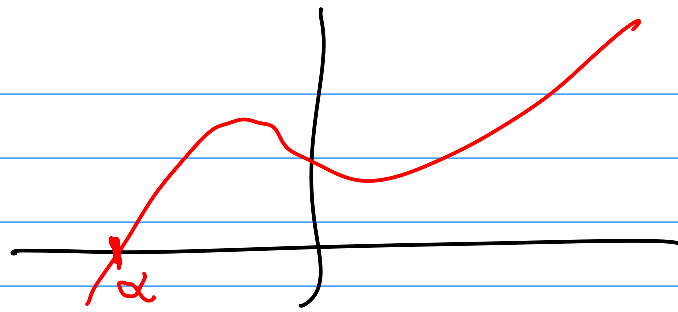
hypothesis on x_0 :

hypothesis on f :

- f need to be cont., in fact, it needs to be differentiable...
- x_0 needs to be close the the ☑ actual solution one is looking for.
- $f'(x_n) \neq 0$ for all n
- f'' exist and is cont

Idea $f'(x) \neq 0$ in a neighborhood of the root you're looking for.

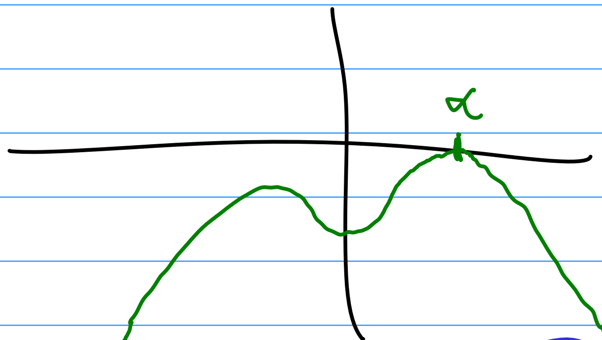
Maybe this. If $f(x)=0$ for $x=\alpha$ Then we want $f'(\alpha) \neq 0$. ☑



If α is the root the graph crosses the x axis at α. which means $\boxed{f'(\alpha) \neq 0.}$

hypothesis...

Not alway like this



Oh no!

$f'(\alpha) = 0$

Hypothesis: $f'(\alpha) \neq 0$ $\quad$ f has a cont derivative and x_0 starts out close to α.

Then solving for the root $x = \alpha$ where $f(\alpha) = 0$ using Newton's method is a good idea...

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

← Calculus

Best theorem in Calculus for analyzing numerics is Taylor's Theorem...

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$\sin x = ?$$

$$\cos x = ?$$

$$\arctan x = ?$$

$$\ln(1+x) = ?$$

$$\frac{1}{1-x} = ?$$

$$(1-x)^\alpha = ?$$

Remember for the weekend

Best theorem in Calculus for analysing numerics is Taylor's Theorem...

Taylor's Theorem: Suppose f has $n+1$ cont. derivatives, Then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$\cdots + \frac{h^n}{n!}f^{(n)}(x) + \boxed{\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(c)}$$

for some value of c between x and $x+h$.

error term allows us to estimate errors.

Sketch of the proof:

$$f(x+h) - f(x) = \int_x^{x+h} f(t)\,dt$$ and then integrate by parts

$$= h\cancel{f'}(x) + \frac{h^2}{2}\cancel{f''}(x) + \frac{h^3}{3!}\cancel{f'''}(x) + \dots$$
$$\dots + \frac{h^n}{n!}\cancel{f^{(n)}}(x) + \text{integral}$$

look up this proof over the weekend

$$\text{integral} = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

using mean value theorem for integrals

$$f(x+h) = f(x) + h\cancel{f'}(x) + \frac{h^2}{2}\cancel{f''}(x) + \frac{h^3}{3!}\cancel{f'''}(x) + \dots$$
$$\dots + \frac{h^n}{n!}\cancel{f^{(n)}}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to make the derivatives the same, set $\boxed{x = x_n}$ in Taylor theorem. (or not)

This would work...

other way set $\boxed{x = \alpha}$ in Taylor theorem...

dividing by $f'(\alpha)$ is good....

Estimate does $x_n \to \alpha$?

Let $e_n = x_n - \alpha$ (signed error in x_n)

How the error change per iteration?

$$e_{n+1} = x_{n+1} - \alpha = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)} - \alpha}$$

$$= e_n - \boxed{\frac{f(x_n)}{f'(x_n)}} \quad \text{now what...}$$

Try $\boxed{\begin{array}{c} x = x_n \\ x+h=\alpha \end{array}}$ in Taylor

solve for h

$h = \alpha - x = \alpha - x_n$

$h = -e_n$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\dots + \frac{h^n}{n!} f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

α is solution

$$0 = f(\alpha) = f(x_n) + (\alpha - x_n) \underbrace{f'(x_n)} + \frac{(\alpha - x_n)^2}{2} f''(c)$$

divide by this

$$0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{(\alpha - x_n)^2}{2} \frac{f''(c)}{f'(x_n)}$$

$$-\frac{f(x_n)}{f'(x_n)} = -e_n + \frac{e_n^2}{2} \frac{f''(c)}{f'(x_n)}$$

$$e_{n+1} = \cancel{e_n} - \cancel{e_n} + \frac{e_n^2}{2} \frac{f''(c)}{f'(x_n)}$$

$$e_{n+1} = \frac{e_n^2}{2} \frac{f''(c)}{f'(x_n)}$$

big enough

Let $A = \max\{|f''(c)| : c \in I\}$

$$B = \min\{|f'(x)| : \underbrace{|x-\alpha| < \varepsilon}\}$$

all the bigger thats needed...

$$e_{n+1} \lesssim \frac{A}{2B} e_n^2$$

doubles the # of sig. digits per iteration...