New review items for the final

- State the power method for finding the eigenvector corresponding to the (unique) eigenvalue of largest magnitude for a matrix A (Oct 7 and 19) ☑
- Given the eigenvalues of $B=A^TA$ be able to compute the matrix 2 norm of A (Nov 9) ☑
  max $\{\sqrt{\lambda_i} : \ldots\}$
- Be able to find an interpolating polynomial through given points (x_i, y_i) using the Lagrange basis functions (Nov 18) (Don't simplify) ... ☑
- Proof of the polynomial interpolation theorem (Nov 16 and 18) ☑
- Relationship between the QR and Cholesky factorization (Homework 2)
- Analysis of the matrix $A=uv^T$ where $u^Tv=1$ (Homework 2)

EXTRA CREDIT: Why does the power method work?

EXTRA CREDIT: Given the eigen vector and eigenvalues of the matrix $B=A^TA$ find the SVD of $A=V\Sigma U^T$ I.e. find V, Σ and U.

Recall:

$$l_k(t) = \prod_{i\neq k} \frac{t-x_i}{x_k-x_i}$$

- polynomial of degree n-1
- $l_k(x_k) = \prod_{i\neq k} \frac{x_k-x_i}{x_k-x_i} = 1$
- $j\neq k$, $l_k(x_j) = \prod_{i\neq k} \frac{x_j-x_i}{x_k-x_i} = 0$

Thus

$$\boxed{l_k(x_j) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}}$$

$i=1$ $\quad$ $i=2$ $\quad$ $i=3$

$(2,5)$ $\quad$ $(4,7)$ $\quad$ $(-3,8)$

$$l_1(t) = \prod_{i\neq 1} \frac{t-x_i}{x_1-x_i} = \frac{(t-x_2)(t-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(t-4)(t+3)}{(2-4)(2+3)} = \frac{(t-4)(t+3)}{-10}$$

$$l_2(t) = \frac{(t-2)(t+3)}{(4-2)(4+3)} = \frac{(t-2)(t+3)}{14}$$

$$l_3(t) = \frac{(t-2)(t-4)}{(-3-2)(-3-4)} = \frac{(t-2)(t-4)}{35}$$

$$p(t) = y_1 l_1(t) + y_2 l_2(t) + y_3 l_3(t)$$

$$p(t) = 5\,\frac{(t-4)(t+3)}{-10} + 7\,\frac{(t-2)(t+3)}{14} + 8\,\frac{(t-2)(t-4)}{35}$$

don't simplify this any more (for the exam).

Review of Polynomial interpolation theorem

Theorem: Let $p(t)$ be the interpolating polynomial of degree $n-1$ such that $p(x_i) = f(x_i)$ for $i=1,\dots,n$. Assume the x_i's are different, also that f has n continuous derivatives. Then

$$f(t) = \underbrace{p(t)}_{\text{interpolating polynomial}} + \frac{q(t)}{n!} f^{(n)}(\xi) \quad \leftarrow E(t)$$

another polynomial

where

$$q(t) = (t-x_1)(t-x_2)\cdots(t-x_n)$$

has some roots as $E(t)$...

and ξ is between $\min(t, x_1, x_2, \dots, x_n)$ and $\max(t, x_1, x_2, \dots, x_n)$.

$$E(t) = f(t) - p(t)$$

$$F(t) = E(t) - \alpha\, q(t)$$

add a root at the location where I want to bound the error

$$F(t_*) = E(t_*) - \alpha\, q(t_*) = 0, \qquad \alpha = \frac{E(t_*)}{q(t_*)}$$

Then F has $n+1$ roots, the roots at the x_i's and the root at t_*.

By Rolls theorem, $F^{(n)}$ has 1 root left and that root is between the original roots..

Thus $F^{(n)}(\xi) = 0$ where ξ is between $\min(t_*, x_1, \dots, x_n)$ and $\max(t_*, x_1, \dots, x_n)$.

$$F^{(n)}(t) = E^{(n)}(t) - \alpha \underbrace{g^{(n)}(t)}_{n!}$$

g is poly of degree n
and $g^{(n)}(t) = \text{const}$
$= n!$

$$E^{(n)}(\xi) - \alpha n! \approx 0$$

$$E^{(n)}(\xi) \approx \frac{E(t_*)}{g(t_*)} n!$$

$$\boxed{E(t_*) = \frac{g(t_*)}{n!} E^{(n)}(\xi) \approx \frac{g(t_*)}{n!} f^{(n)}(\xi)}$$

Since

$$E^{(n)}(t) = f^{(n)}(t) - \underbrace{p^{(n)}(t)}_{0}.$$

interpolating polynomial of degree $n-1$