Recall the graph of

$$f(x) = \frac{e^x - 1}{x} - 1 \quad \text{when } x \text{ was small}$$

The problem is loss of precision due to subtraction of nearly equal numbers.

Whats the solution:

Consider two different cases

① $f(x)$ when x is close to zero

② $f(x)$ when x is NOT close to zero...

In the ② case just use the formula

$$f(x) = \frac{e^x - 1}{x} - 1$$

and plug in x on the computer

What to do in case ①:

Usual Taylor series for e^x works really well when x is close to zero...

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$\approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$ to at least 15 digits of precision when x is close to zero...

$(10^{-5})^3 = 10^{-15}$ so 15 digits

size of x for this approx. to be good...

Idea: rewrite $f(x)$ in terms of Taylor series when x is small...

$$\frac{e^x - 1}{x} - 1 \approx \frac{\cancel{1} + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \cancel{1}}{x} - 1$$

$$\approx \frac{\cancel{x} + \frac{1}{2!}x^{\cancel{2}} + \frac{1}{3!}x^{\cancel{3}\,2}}{\cancel{x}} - 1$$

$$\approx \cancel{1} + \frac{1}{2}x + \frac{1}{3!}x^2 - \cancel{1}$$

$$\approx \frac{1}{2}x + \frac{1}{3!}x^2$$

HW 25

Remember the Taylor series for natural log?

Start with the geometric series:

$$1 + x + x^2 + x^3 + x^4 + \cdots =$$

In general

recall

$$.33333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

$$= 3\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{3}{10}\left(\overbrace{1 + x + x^2 + \cdots}^{10/9}\right)$$

where $x = \frac{1}{10}$.

$$= \frac{1}{3}$$

$$s = .3\cancel{3}\cancel{3}\cancel{3}\cancel{3}\ldots$$

$$\frac{1}{10}s = .0\cancel{3}\cancel{3}\cancel{3}\cancel{3}\ldots$$

$$\left(1 - \frac{1}{10}\right)s = .3$$

$$\frac{9}{10}s = .3, \quad s = \frac{1}{3}$$

$$S = 1 + \cancel{x} + \cancel{x^2} + \boxed{x^3} + x^4 + \cdots$$

$$xS = \cancel{x} + \cancel{x^2} + \boxed{x^3} + x^4 + x^5 + \cdots$$

$$(1-x)S = 1$$

$$S = \frac{1}{1-x}$$

recall $\log t = \int_1^t \frac{1}{x}\,dx$

$$1 + x + x^2 + x^3 + x^4 + \cdots = \underbrace{\frac{1}{1-x}}$$

integrate this

shift by 1 already so integrate from 0 to t.

$$\int_0^t \left(1 + x + x^2 + x^3 + x^4 + \cdots\right) dx = \int_0^t \frac{1}{1-x}\,dx$$

$$t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5 + \cdots = -\log(1-t)$$

$$\int_0^t \frac{1}{1-x}\,dx = -\int_1^{1-t} \frac{1}{u}\,du = -\log(1-t)$$

$u = 1-x$

$du = -dx$

set $x = -t$

$t = -x$

$$-x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 + \cdots = -\log(1+x)$$

Finally

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$

$$\text{log1p}(x) = \log(1+x)$$

```
julia> pi
π = 3.1415926535897...

julia> pi/1e9
3.141592653589793e-9

julia> 1+pi/1e9
1.0000000031415928

julia> log(1+pi/1e9)
3.141592758776485e-9

julia> log1p(pi/1e9)
3.141592648654991e-9
```

π to 15 significant digits

still 15 significant digits

loss of precision by adding a large number to a small

only 8 digits of π are left...

all these zeros took space to store

inaccurate

since $\log(1) = 0$ then a number close to 1 gets mapped to a number close to zero. This is like subtracting 1 but we do it with logarithms...

numbers only agree to 7 digits...

This is the accurate answer

2.5 Suppose we are given a list of floating-point values $x_1, x_2, \ldots, x_n$. The following quantity, known as their "log-sum-exp," appears in many machine learning algorithms:

$$\ell(x_1, \ldots, x_n) \equiv \ln\left[\sum_{k=1}^{n} e^{x_k}\right].$$

(a) The value $p_k \equiv e^{x_k}$ often represents a probability $p_k \in (0, 1]$. In this case, what is the range of possible x_k's?

This sum contains two types of terms... e^{x_k} close to zero an e^{x_k} not close to zero

Idea write

$$\sum_{k=1}^{n} e^{x_k} = \sum \text{large terms} + \sum \text{small terms}$$

and then rescale by as

$$\frac{\sum \text{large terms} + \sum \text{small terms}}{\sum \text{large terms}} = \underbrace{1 + \frac{\sum \text{small terms}}{\sum \text{large terms}}}$$

and then use log1p to compute the log of this.

(a) Thanks to inaccuracies in how we evaluate or express $f(x)$, we might accidentally compute roots of a perturbation $f(x) + \varepsilon p(x)$. Take x^* to be a root of f, so $f(x^*) = 0$. If $f'(x^*) \neq 0$, for small ε we can write a function $x(\varepsilon)$ such that $f(x(\varepsilon)) + \varepsilon p(x(\varepsilon)) = 0$, with $x(0) = x^*$. Assuming such a function exists and is differentiable, show:

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{p(x^*)}{f'(x^*)}.$$

use implicit differentiation

$$\boxed{f(x(\varepsilon)) + \varepsilon p(x(\varepsilon)) = 0}$$

As I change the problem how does the solution change?. the solution changes as $\frac{dx(\varepsilon)}{d\varepsilon}$ ..

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NEXT TOPIC: QR factorization....