Ways of measuring errors: $x \in \mathbb{R}$ is the exact value and x^* the approximation

① absolute error $\quad e_{abs} = |x - x^*|$

② relative error ... $\quad e_{rel} = \dfrac{e_{abs}}{|x|}$

Types:

approximation	approximand	type of error
v^*	v	initial error
$M(v^*)$	$M(v)$	propagated error
$\boxed{M^*(v^*)}$	$\boxed{M(v^*)}$	generated error
$M^*(v^*)$	$M(v)$	total cumulative error
what we get	what we want	

M is the function or algorithm we want to apply.

M* is an approximation of that function programmed into a computer..

```
julia> 4.27*3.68
15.7136
```

already initial error in the inputs

add the exponents

$$(4.27 \times 10^1) \times (3.68 \times 10^1) = \underbrace{15.7136}_{} \times 10^2 \rightarrow 1.57 \times 10^3$$

rounding step 3S generates more error..

```
julia> 4.27e1*3.68e1
1571.36
```

round back to 3S

given $1570. = 1.57 \times 10^3$

Because of the rounding the computer doesn't perform multiplication exactly...and so ...

Addition of a small to a large # may have no effect..

$$5.18 \times 10^2 + 4.37 \times 10^{-1} = 5.18 \times 10^2 + 0.00437 \times 10^2$$
$$= 5.18437 \times 10^2 \rightarrow 5.18 \times 10^2$$

How much smaller does the have to be for no effect?

5.18×10^2
$+\ 4.37\times10^{-1}$

$$\begin{array}{rr} & 5.18 \quad \times10^2 \\ + & 0.00437 \quad \times10^2 \\ \hline & 5.18437\times10^2 \end{array}$$

exact sum

generated error

$\hookrightarrow$ rounding 3S: 5.18×10^2

same number...

when only 3 significant digits this seems extreme...but can still happen on a real computer...

What is the smallest value of $\varepsilon>0$ such that $\boxed{1+\varepsilon\neq1}$?

```
julia> 1+0.0001
1.0001

julia> 1+0.0001==1
false
```

From this I conclude that ε, whatever it is, must be smaller than 0.0001

```
julia> epsilon=0.000000001
1.0e-9

julia> 1+epsilon==1
false

julia> epsilon=0.0000000000000000000000000000001
1.0e-31

julia> 1+epsilon==1
true
```

← still large enough to make a difference...

← ε is so small that adding it to 1 does nothing..

Note the value of epsilon just large enough that it does something it called Machine epsilon...

```
julia> epsilon=1e-16
1.0e-16

julia> 1+epsilon==1
true

julia> epsilon=1e-15
1.0e-15

julia> 1+epsilon==1
false
```

too small

makes a difference ..

1.0
0.000000000000001

indicates about 15 significant digits are available ..

also:

the associative law for addition does not always hold.

$$(a+b)+c = a+(b+c)$$

If approximate addition... it can happen that

$$(a +^* b) +^* c \neq a +^* (b +^* c)$$

```
julia> a=1.0
1.0

julia> b=2.0
2.0

julia> c=3.0
3.0

julia> (a+b)+c==a+(b+c)
true
```

In this case associativity holds...

```
julia> a=0.1
0.1

julia> b=0.2
0.2

julia> c=0.3
0.3

julia> (a+b)+c==a+(b+c)
false
```

there is actually initial error here...

$a \approx 0.1$ but not exact because working in base 2...

Non associative, because of the rounding errors...

```
julia> (a+b)+c
0.6000000000000001

julia> a+(b+c)
0.6
```

this looks strange because the computer is working base 2 behind the scenes...

Scientific notation base 10. $.1 = 1.0 \times 10^{-1}$

if this were

$.1 = m \times 2^n$

never get a factor of 10 from powers of 2...

this simple number in decimal has a repeating binary expansion...

any fraction that doesn't have a power of 2 in the denominator turns into a repeating binary expansion.

recall for decimal numbers, fractions of the form

$$\frac{p}{2^n 5^m} \quad \text{all terminate}$$

but not the others...

- Step 01
- Step 02
- Step 03, Non-Associativity of Addition, Types of Errors
- Step 04, Realistic Floating Point Data Types, Rounding Examples
- Step 05, Taylor's Theorem

read to here for wednesday...

already initial error in the inputs

add the exponents

```
julia> 4.27*3.68
15.7136
```

$$(\underbrace{4.27}_{a^*} \times 10^1) \times (\underbrace{3.68}_{b^*} \times 10^1) = 15.7136 \times 10^2 \to 1.57 \times 10^3$$

generated...

$e_{rel}(a) \le 5\times10^{-3}$

$e_{rel}(b) \le 5\times10^{-3}$ → propagated error ≈ is sum of rel. error in multiplication

$$\boxed{e_{rel}(a+b) \le 10\times10^{-3} = 1\times10^{-2}}$$