Using division to evaluate a polynomial...

Let $p(x)$ be a polynomial.

Want to find $p(3)$?

divisor degree $=1$

$$(x-3)\overline{)\,p(x)\,}^{\,q(x)} \quad + \text{ remainder } r(x)$$

one degree less than the divisor degree $=0$

Thus $p(x) = (x-3)\,q(x) + r$ constant...

$$p(3) = \cancel{(3-3)}^{0}\,q(3) + r = r$$

don't need to plug 3 into q.

- Use synthetic division to find r and this is the computationally efficient way to find $p(3)$.
- Not only is this efficient but it minimizes rounding errors...

Good Algorithm $\left\{ \begin{array}{l} \text{both efficient} \\ \text{and accurate...} \end{array} \right.$

Where do polynomial approximations come from?

One place they come from: Taylor's theorem...

$$f(x) = \underbrace{\boxed{\text{Taylor Polynomial}}}_{\text{approximation}} + \underbrace{\text{Remainder term}}_{\text{error in approximation}}$$

Theorem because it says something about the remainder...

Taylor's theorem:

Fundamental Theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

Let $b = x+h$
$a = x$

$$f(x+h) - f(x) = \int_x^{x+h} f'(t)\,dt$$

0^{th} order Taylor theorem

$$\underbrace{f(x+h)}_{\text{function of } h} = \underbrace{f(x)}_{\text{approximation}} + \underbrace{\int_x^{x+h} f'(t)\,dt}_{\text{remainder}}$$

Integrate by parts to construct a higher order Taylor approximation...

$$\int_x^{x+h} f'(t)\,dt = uv\Big|_x^{x+h} - \int_x^{x+h} v\,du$$

$u = f'(t)$ $\qquad du = f''(t)\,dt$

$dv = dt$ $\qquad v = t + c$

tricky to put constant c in here, well use it to simplify later.

$$= f'(t)(t+c)\Big|_x^{x+h} - \int_x^{x+h} (t+c) f''(t)\,dt$$

two terms...choose c so one goes away...

$$= \underbrace{f'(x+h)(x+h+c)} - f'(x)(x+c) - \int_x^{x+h} (t+c) f''(t)\,dt$$

make this go away... set $c = -x-h$

$$= -f'(x)\left(\cancel{x} + (-\cancel{x} - h)\right) - \int_x^{x+h} (t-x-h) f''(t)\,dt$$

Therefore...

$$\int_x^{x+h} f'(t)\,dt = h f'(x) + \int_x^{x+h} (x+h-t) f''(t)\,dt$$

Since

$$\underbrace{f(x+h)}_{\text{function of } h} = \underbrace{f(x)}_{\text{approximation}} + \underbrace{\int_x^{x+h} f'(t)\,dt}_{\text{remainder}}$$

First order Taylor approximation

$$\underbrace{f(x+h)}_{\text{function of } h} = \underbrace{f(x) + h f'(x)}_{\text{approximation}} + \underbrace{\int_x^{x+h} (x+h-t) f''(t)\, dt}_{\text{remainder...}}$$

To get 2^{nd} order Taylor Approximation integrate by parts again...

and again...