Fundamental Theorem of Calculus:

$$f(x+h) - f(x) = \int_x^{x+h} f'(t)\,dt$$

Integrate by parts

$u = f'(t)$ $\quad$ $du = f''(t)dt$

$dv = dt$ $\quad$ $v = t \underbrace{- x - h}$

this does not depend on t so const. with respect to the integral..

$$= uv\Big|_x^{x+h} - \int_x^{x+h} v\,du$$

cancels out

$$= f'(t)(t-x-h)\Big|_x^{x+h} - \int_x^{x+h} (t-x-h) f''(t)\,dt$$

only this is left

$$= -f'(x)(x-x-h) + \int_x^{x+h} (x+h-t) f''(t)\,dt$$

$$= \underbrace{hf'(x)}_{\text{first order term in Taylor polynomial..}} + \int_x^{x+h} \underbrace{(x+h-t) \underbrace{f''(t)}_{u}\, dt}_{dv}$$

another integration by parts

$u = f''(t)$ $\quad$ $du = f'''(t)dt$

$dv = (x+h-t)dt$ $\quad$ $v = -\frac{1}{2}(x+h-t)^2$

$$= hf'(x) + uv\Big|_x^{x+h} - \int_x^{x+h} v\,du$$

cancels

$$= hf'(x) + f''(t)\left(-\frac{1}{2}(x+h-t)^2\right)\Bigg|_x^{x+h} + \int_x^{x+h} \frac{1}{2}(x+h-t)^2 f'''(t)\,dt$$

only lower endpoint is left

$$= hf'(x) + \frac{h^2}{2}f''(x) + \int_x^{x+h} \frac{1}{2}(x+h-t)^2 f'''(t)\,dt.$$

In summary

$$f(x+h) = \underbrace{f(x) + hf'(x) + \frac{h^2}{2}f''(x)}_{\substack{\text{Taylor Polynomial (in } h) \\ \text{(approximation)}}} + \underbrace{\int_x^{x+h} \frac{1}{2}(x+h-t)^2 f'''(t)\,dt}_{\substack{\text{remainder} \\ \text{(error)}}}.$$

What's the pattern?

$$\boxed{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x) + \int_x^{x+h} \frac{1}{n!}(x+h-t)^n f^{(n+1)}(t)\,dt}$$

- Taylor's Theorem with integral form of the remainder...
- What were the hypothesis in this theorem?

  For example, this works if f has $n+1$ continuous derivatives...

Theorem: Suppose f has $n+1$ continuous derivatives then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \int_x^{x+h} \frac{1}{n!}(x+h-t)^n f^{(n+1)}(t)\, dt$$

Lagrange form of the remainder in Taylor's theorem...

$$R = \int_x^{x+h} \frac{1}{n!}(x+h-t)^n f^{(n+1)}(t)\, dt$$

Think of this as an weighted average of $f^{(n+1)}(t)$.

Mean value theorem for integrals:

Idea: The (weighted) average lies somewhere between the max and the min.

assumed that $h>0$

weight in the weighted average...

$$\frac{\displaystyle\int_x^{x+h} \boxed{\frac{1}{n!}(x+h-t)^n} f^{(n+1)}(t)\, dt}{\displaystyle\int_x^{x+1} \frac{1}{n!}(x+h-t)^n\, dt}$$

now that's really a weighted average...

divide by the sum of the weights to make a weighted average

$$\min f^{(n+1)}(t) \leq \boxed{\text{weighted average of } f^{(n+1)}(t)} \leq \max f^{(n+1)}(t)$$

Simplify the denominator...

$$\int_x^{x+h} \frac{1}{n!}(x+h-t)^n\,dt = \frac{-1}{(n+1)!}(x+h-t)^{n+1}\Big|_x^{x+h} = \frac{h^{n+1}}{(n+1)!}$$

Cancels

only this endpoint left

$$\frac{h^{n+1}}{(n+1)!}\min f^{(n+1)}(t) \leq \underbrace{\int_x^{x+h} \boxed{\frac{1}{n!}(x+h-t)^n} f^{(n+1)}(t)\,dt}_{R} \leq \frac{h^{n+1}}{(n+1)!}\max f^{(n+1)}(t)$$

$R \leftarrow$ between max and min

Since a continuous function takes on all values between its max and min, then there is a point ξ between x and $x+h$ such that

$$R = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

$\leftarrow$ Lagrange form of the remainder.

Theorem: Suppose f has $n+1$ continuous derivatives then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

for some value of ξ between x and $x+h$.