Step 8

Analysis of the rate of convergence of the secant method...

$$e_{n+1} = \alpha - x_{n+1}$$

$$e_{n+1} = \frac{e_{n-1}[f(\alpha) - e_n f'(\alpha) + (e_n^2/2!) f''(\alpha) - \cdots]}{[f(\alpha) - e_n f'(\alpha) + \cdots] - [f(\alpha) - e_{n-1} f'(\alpha) + \cdots]}$$

$$- \frac{e_n[f(\alpha) - e_{n-1} f'(\alpha) + (e_{n-1}^2/2!) f''(\alpha) - \cdots]}{[f(\alpha) - e_n f'(\alpha) + \cdots] - [f(\alpha) - e_{n-1} f'(\alpha) + \cdots]}$$

$$\approx - \left[\frac{f''(\alpha)}{2 f'(\alpha)} \right] e_{n-1} e_n$$

Thus $\boxed{e_{n+1} \simeq K e_{n-1} e_n}$

Rate of convergence for bisection method...

$e_1 = \frac{1}{2}|b-a|$ $[a, b]$; $c = \frac{a+b}{2}$

error is proportional to length of interval

new interval either $[a, c]$ or $[c, b]$

$e_2 = \frac{1}{4}|b-a|$

In general

$$e_n = \frac{1}{2^n} |b-a|$$

Theoretically we consider the number of times we have to evaluate f as the deciding factor in efficiency of root finding...

Because f may be a complicated program that takes a long time to run...

In general

$$e_n = \frac{1}{2^n} |b-a|$$

The error seems to be decaying exponentially

Actually this type of convergence is not that fast compared to doubling the number of significant digits at each iteration.

Another way of characterizing this convergence rate is

$$e_{n+1} \approx \frac{1}{2} e_n$$

this is a linear relation and we call this linear convergence...

What kind of convergence doubles the number of significant digits?

$$e_{n+1} \simeq K e_n^2$$

quadratically convergent.

Note if $e_n = 10^{-m}$ then

this m is like the # of significant digits

$$e_{n+1} \approx K e_n^2 \approx K(10^{-m})^2$$

$$\approx K 10^{-2m}$$

twice as many

Secant method has

$$e_{n+1} \approx K e_{n-1} e_n$$

$e_n \approx M e_{n-1}^k$ shifted version of

Suppose this is the same as

$$\boxed{e_{n+1} \approx M e_n^k}$$

plug it in

and look for a k that makes things consistent

$$M e_n^k \approx K e_{n-1} M e_{n-1}^k$$

$$M(M e_{n-1}^k)^k \approx K e_{n-1} M e_{n-1}^k$$

Write this again in case difficult to follow the first time...

So from the beginning...

Secant Method

want to find the power k so that

$$e_{n+1} \approx M e_n^k$$

also $e_n \approx M e_{n-1}^k$

something about the secant method...

what do I know?

$$e_{n+1} \approx K e_{n-1} e_n$$

Idea substitute and solve for k.

$$M e_n^k \approx K e_{n-1} e_n$$

$$M\left(M e_{n-1}^k\right)^k \approx K e_{n-1} M e_{n-1}^k$$

$$M M^k e_{n-1}^{k^2} \approx K M e_{n-1}^{k+1}$$

These two terms need to be the same for consistency

Thus $k^2 = k+1$ now solve for k

$$K^2 - K - 1 = 0$$

$a = 1 \quad b = -1 \quad c - 1$

so k is positive

$$K = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$$

Golden ratio

Secant method

$$e_{n+1} \leq K e_n^{\frac{1+\sqrt{5}}{2}} \approx K e_n^{1.618}$$

Superlinear convergence...