- Step 9 Ex
- Step 10 E

Focus on the idea of iterative approximation..,.

Suppose x_1 is an initial approximation then apply an algorithm represented by φ to obtain an "improved" approximation $x_2 = \varphi(x_1)$

φ: approximation as the input $\longrightarrow$ improved approximation output.

Thus
$$x_2 = \varphi(x_1)$$
$$x_3 = \varphi(x_2)$$
$$x_4 = \varphi(x_3)$$
$$\vdots$$
$$x_{n+1} = \varphi(x_n)$$

one step algorithm becomes only the last approx. is used to create the next one....

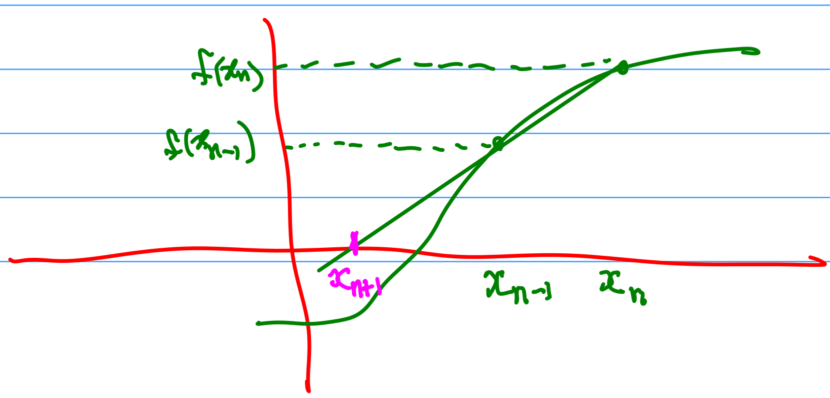
- mathematically we hope to show $x_n \to$ the correct answer as $n \to \infty$.

Sometimes we think of a multistep algorithm

$$x_{n+1} = \varphi(x_n, x_{n-1})$$ two steps...

Secant method...

$$x_{n+1} = \varphi(x_n, x_{n-1}, x_{n-2})$$ three step...



Eq. of line passing through two points.

Point-Slope form: $y - f(x_n) = m\,(x - x_n)$

$$m = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Eq of the line....

$$y - f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\,(x - x_n)$$

Find where line intersects x-axis: $y = 0$

$$-f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\,(x_{n+1} - x_n)$$

$$x_{n+1}\,\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = -f(x_n) + x_n\,\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \phi(x_n, x_{n-1})$$

where $\phi(a,b) = a - f(a)\,\dfrac{a-b}{f(a) - f(b)}$

Secant method written as a two step algorithm to improve the approximation...

Multistep methods involve more history of the approximations and that can give an advantage, but this comes at the cost of additional complexity both in the analysis and implementation of the algorithm.

Single step methods: $x_{n+1} = g(x_n)$

When does this converge? If so to what?

$$x_{n+1} = g(x_n)$$

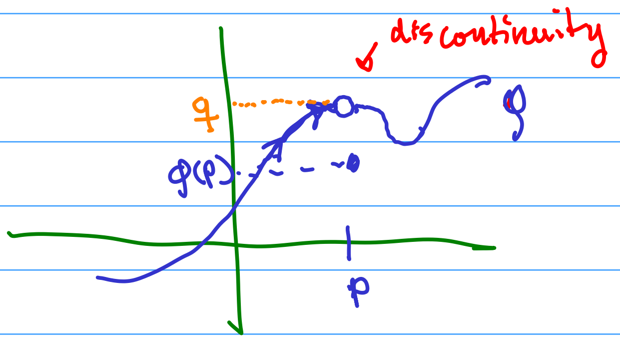
assume $x_n \to p$ as $n \to \infty$.

as $n \to \infty$

$$\boxed{\underbrace{p = g(p)}_{\text{a point } p \text{ like this is called a fixed point of } g.}}$$

p is a fixed point of g ... provided the function g is continuous

Continuity in Calculus



$$\lim_{x \to p} g(x) = q$$

$$g(p) \neq q$$

In what follows we'll assume g is continuous and even that it has continuous derivatives...

$$p - x_2 = g(p) - g(x_1) = g'(\xi_1)(p - x_1)$$

for some ξ_1 between p and x_1

Define $e_n = p - x_n$ then

$e_2 = \varphi'(\xi_1)\, e_1$

for some ξ_2 between p and x_2.

$$e_3 = p - x_3 = \varphi(p) - \varphi(x_2) = \varphi'(\xi_2)(p - x_2) = \varphi'(\xi_2)\, e_2$$

plug it in $\quad e_3 = \varphi'(\xi_1)\varphi'(\xi_2)\, e_1$

Following the pattern...

$$e_{n+1} = \varphi'(\xi_n)\, e_n \qquad |e_{n+1}| \leq |\varphi'(\xi_n)|\,|e_n|$$

$$e_{n+1} = \varphi'(\xi_1)\varphi'(\xi_2)\cdots\varphi'(\xi_n)\, e_1$$

$$|e_{n+1}| = |\varphi'(\xi_1)||\varphi'(\xi_2)|\cdots|\varphi'(\xi_n)|\,|e_1|$$

$$|e_{n+1}| = \left(\prod_{k=1}^{n} |\varphi'(\xi_k)|\right)|e_1|$$

If $|\varphi'(\xi_k)| < \alpha < 1$ for all k's then ...

$$|e_{n+1}| \leq \left(\prod_{k=1}^{n} \alpha\right)|e_1| = \alpha^n |e_1|$$

since $0 \leq \alpha < 1$ then $\alpha^n \to 0$ as $n \to \infty$.

Therefor $x_n \to p$ as $n \to \infty$

Condition for convergence: if $|\varphi'(\xi)| < 1$ for all values of ξ close to the fixed point p.

Note if x_1 is close enough to p such that $|\varphi'(\xi)| \leq \alpha < 1$ for all ξ between p and x_1 then $|\varphi'(\xi_k)| \leq \alpha$ for all k ...

and so $x_{n+1} = \varphi(x_n)$ converges