Newton's Method

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

Newton's method is a fixed-point method...

Thus

$$x_2 = \varphi(x_1)$$
$$x_3 = \varphi(x_2)$$
$$x_4 = \varphi(x_3)$$
$$\vdots$$
$$x_{n+1} = \varphi(x_n)$$

hope $x_n \to$ solution as $n \to \infty$

Fixed point method means problem I trying to solve is given by $x = \varphi(x)$

$$\varphi(x) = x - \frac{f(x)}{f'(x)}.$$

what does $\varphi(x) = x$ mean? plug in and find out...

since $x - \dfrac{f(x)}{f'(x)} = x$

then $-\dfrac{f(x)}{f'(x)} = 0$

then $-f(x) = 0$

or $f(x) = 0$

So the only way $\varphi(x) = x$ is when $f(x) = 0$ that's the problem we're trying to solve...

From last Wednesday.

If $|g'(\xi_k)| < \alpha < 1$ for all k's then ...

$$|e_{n+1}| \leq \left(\prod_{k=1}^{n} \alpha\right)|e_1| = \alpha^n |e_1|$$

since $0 \leq \alpha < 1$ then $\alpha^n \to 0$ as $n \to \infty$.

So what $g'(x)$ for Newton's method?

$$g(x) = x - \frac{f(x)}{f'(x)}$$

quotient rule...

$$g'(x) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right) = 1 - \frac{\underbrace{f'(x)f'(x)}_{f'(x)^2} - f(x)f''(x)}{f'(x)^2}$$

$$= \cancel{1} - \cancel{1} + \frac{f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

We want $|g'(x)| < 1$.

$$\left|\frac{f(x)f''(x)}{f'(x)^2}\right| < 1$$

we know that if p is the answer than $f(p) = 0$

Thus if x is close to p then $f(x)$ is close to $f(p) = 0$.

Therefore, if x_1 the initial approximation is such that

here any number less that 1 works for α.

$$|g'(x)| = \boxed{\left|\frac{f(x)f''(x)}{f'(x)^2}\right|} \leq \alpha < 1 \quad \text{for all} \quad |x - p| \leq |x_1 - p|$$

all points as close or closer to p as x_1

Then $|g'(x)| \leq \alpha < 1$ for every x_k that's subsequently generated by the iteration.

Note that since

$$\text{error} \rightarrow |e_{n+1}| \leq \left(\prod_{k=1}^{n} \alpha\right) |e_1| = \alpha^n |e_1|$$

then the smaller α is the faster the error goes to zero

On the other hand larger values of α are less restrictive for the starting approximation x_1.

Since $|\varphi'(x)| \to 0$ as $x_n \to p$ then the actual values of α get smaller as the iteration proceeds...,

As $x_n \to p$ the rate at which it converges increases. That is $\alpha \to 0$.

Note also, since $|\varphi'(p)| = 0$ then we know there is some interval about p such that
$|\varphi'(x)| \leq \alpha \; (\alpha < 1)$ on that interval.

Thus if x_1 is close enough to p it converges.

as it converges it converges faster,

How fast does the convergence increase as $x_n \to p$?

So fast the number of sig. digits doubles at each iteration. (next time).