Theorem: Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Let p be a root of f such that $f(p)=0$ and $f'(p) \neq 0$. Suppose x_1 is close enough to p so that for some $\alpha \in [0,1)$ that

$$(*) \quad \left| \frac{f(x) f''(x)}{f'(x)^2} \right| \leq \alpha \qquad \text{for all } x \text{ where } |x-p| \leq |x_1 - p|$$

Then Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges and $\lim_{n\to\infty} x_n = p$.

Remark: If we know $f'(p) \neq 0$ then guaranteed there is and x_1 close enough to p and an $\alpha \in [0,1)$ for which the condition $(*)$ is satisfied.

Now that we know when Newton's method converges, the question is how fast?

Integral Remainder form of Taylor's theorem...

Theorem: Suppose f has $n+1$ continuous derivatives then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x)$$

$$+ \int_x^{x+h} \frac{1}{n!} (x+h-t)^n f^{(n+1)}(t)\, dt$$

Lagrange Remainder form of Taylor's theorem:

Theorem: Suppose f has $n+1$ continuous derivatives then

stop with $n=1$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x)$$
$$+ \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

← remainder involved the second derivative...

for some value of ξ between x and $x+h$.

Recall Newtons method is the fixed point iteration given by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

We assume there is a solution $f(p)=0$ with $f'(p) \neq 0$ and all the other stuff needed so that $x_n \to p$ as $n \to \infty$.

Given x_n the curret approximation $x_{n+1} = g(x_n)$

p the solution we have $p = g(p)$

Two choices for Taylor's theorem

$x+h = x_n$ and $x = p$

or

$x+h = p$ and $x = x_n$ ← book uses this option...

$h = p - x = p - x_n = -e_n$

Plug into Taylors Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots$$
$$+ \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \leftarrow$$

Thus

divide by this to get this → $\frac{f(x_n)}{f'(x_n)}$ from Newton's method

$$0 = f(p) = f(x_n) + hf'(x_n) + \frac{h^2}{2}f''(\xi_n)$$

for some ξ_n between p and x_n.

Thus,

$$0 = \frac{f(x_n)}{f'(x_n)} + h + \frac{h^2}{2}\frac{f''(\xi_n)}{f'(x_n)}$$

How does the error change for step to step

$$e_n = x_n - p$$

$$e_{n+1} = x_{n+1} - p = g(x_n) - p$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} - p = e_n - \frac{f(x_n)}{f'(x_n)}$$

$$= e_n + h + \frac{h^2}{2}\frac{f''(\xi_n)}{f'(x_n)}$$

What was h?

$h = p - x = p - x_n = -e_n$

$$= \cancel{e_n} - \cancel{e_n} + \frac{e_n^2}{2}\frac{f''(\xi_n)}{f'(x_n)}$$

Thus $$e_{n+1} = \frac{e_n^2}{2}\frac{f''(\xi_n)}{f'(x_n)}$$

This is the quadratic convergence