Continuing with pivoted LU factorization

$$LU = PA$$

Thus

$$A = P^{-1} LU$$

matrix mult. is composition of linear functions.

parathesis mean compute LU first an the apply P^{-1}.

```
julia> (L*U)[invperm(p),:]
3×3 Matrix{Float64}:
 1.0  2.0  3.0
 4.0  5.0  6.0
 1.0  0.0  1.0
```

$$P^{-1}(L(U(x)))$$

compute the LU first... ie. means apply the functions from right to left...

$PA =$

```
julia> A[p,:]
3×3 Matrix{Int64}:
 4  5  6
 1  0  1
 1  2  3
```

$= \quad LU =$

```
julia> L*U
3×3 Matrix{Float64}:
 4.0  5.0  6.0
 1.0  0.0  1.0
 1.0  2.0  3.0
```

where we did the factorization using the commands

```
julia> A
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 1  0  1

julia> L,U,p=lu(A)
LU{Float64, Matrix{Float64}}
L factor:
3×3 Matrix{Float64}:
 1.0    0.0  0.0
 0.25   1.0  0.0
 0.25  -0.6  1.0
U factor:
3×3 Matrix{Float64}:
 4.0   5.0    6.0
 0.0  -1.25  -0.5
 0.0   0.0    1.2
```

Cholesky factorization:

If $A \in \mathbb{R}^{n\times n}$ and $A^T = A$ (symmetric)

and $\underbrace{x \cdot Ax > 0 \text{ for every } x \in \mathbb{R}^n}_{\text{positive definite}}$

positive definite ☑ means no pivoting need to make LU factorization..

Then there is a triangular matrix L such that

$$A = L L^T$$

L is a lower triangular matrix.

This is like LU factorization where $U = L^T$.

Suppose $A = LU$ Can we find the Cholesky factorization from the LU factorization? Seems similar...

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

```
julia> B=[4 2 1; 2 3 1; 1 1 4]
3×3 Matrix{Int64}:
 4  2  1
 2  3  1
 1  1  4
```

This matrix is symmetric and positive definite.

```
julia> L,U,p=lu(B)
LU{Float64, Matrix{Float64}}
L factor:
3×3 Matrix{Float64}:
 1.0   0.0   0.0
 0.5   1.0   0.0
 0.25  0.25  1.0
U factor:
3×3 Matrix{Float64}:
 4.0  2.0  1.0
 0.0  2.0  0.5
 0.0  0.0  3.625

julia> p
3-element Vector{Int64}:
 1
 2
 3
```

$1 \to 1$, $2 \to 2$, $3 \to 3$

no permutation was used in this case...

Try the LU factorization

$$U = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & .5 \\ 0 & 0 & 3.625 \end{bmatrix}$$

diagonal matrix

$r_1 \leftarrow \frac{1}{4} r_1$

$r_2 \leftarrow \frac{1}{2} r_2$

$r_3 \leftarrow \frac{1}{3.625} r_3$

```
julia> U[1,:]=1/4*U[1,:];
       U[2,:]=1/2*U[2,:];
       U[3,:]=1/3.625*U[3,:];

julia> U
3×3 Matrix{Float64}:
 1.0  0.5  0.25
 0.0  1.0  0.25
 0.0  0.0  1.0
```

← This really is L^T

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3.625 \end{bmatrix}$$

undoes the rescaling

Thus...

$$A = LU = LDL^T$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

same

```
julia> D=diagm([4,2,3.625])
3×3 Matrix{Float64}:
 4.0  0.0  0.0
 0.0  2.0  0.0
 0.0  0.0  3.625

julia> L*D*L'
3×3 Matrix{Float64}:
 4.0  2.0  1.0
 2.0  3.0  1.0
 1.0  1.0  4.0
```

$$\sqrt{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3.625} \end{bmatrix}$$

note $\sqrt{D} = \sqrt{D}^T$

$$A = LU = LDL^T = L\sqrt{D}\sqrt{D}L^T$$

$$= L\sqrt{D}\sqrt{D}^T L^T = (L\sqrt{D})(L\sqrt{D})^T$$

cholesky factor

```
julia> L2=L*sqrt(D)
3×3 Matrix{Float64}:
 2.0  0.0       0.0
 1.0  1.41421   0.0
 0.5  0.353553  1.90394
```

verification

```
julia> L2*L2'
3×3 Matrix{Float64}:
 4.0  2.0  1.0
 2.0  3.0  1.0
 1.0  1.0  4.0
```

So LU factorization can be used to find the Cholesky decomposition. However, some computational efficiency results if we numerically aim for the Cholesky from the beginning...next time