$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

(where we have used the fact that $a_{21} = a_{12}$), we need a decomposition of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{pmatrix},$$

which means that we must have

Solve these

$$\left\{ \begin{aligned} a_{11} &= l_{11}^2 \\ a_{12} &= l_{11} l_{21} \\ a_{22} &= l_{21}^2 + l_{22}^2. \end{aligned} \right. \qquad (2.11)$$

$l_{11} = \sqrt{a_{11}}$ need $a_{11} \geq 0$ (really need... $a_{11} > 0$)

$l_{21} = \frac{a_{12}}{l_{11}}$ need $l_{11} \neq 0$

$l_{22} = \sqrt{a_{22} - l_{21}^2}$ need $a_{22} \geq l_{21}^2$

It appears that we may use the decomposition from the 1×1 case here; using the already known $l_{11} = \sqrt{a_{11}}$ from the 1×1 case, we can solve $a_{12} = l_{11} l_{21}$ for l_{21} and then use it to solve $a_{22} = l_{21}^2 + l_{22}^2$ for l_{22}.

On one hand these tests are all guaranteed to be fine provided $A = A^T$ (symmetric) and $x \cdot Ax > 0$ for all x (positive definite)

On the other hand, these tests can be viewed as a way of checking the A is positive definite.

Where do positive definite matrices come from:

① Elliptic PDE's. — Statics problems, like Stress & Strain ... etc.

because the defining property of being eliptic is the continuous math analogue of A being positive definite. So when you discritize the PDE you immediately get a positive definite A.

This trick is used for the singular value decomposition and also for finding the matrix norm of A.

② Create a positive definite matrix from one that isn't.

Obvious way is to set $B = A^T A$ then B is positive definite (symmetric) provided A is invertible

Why... Need two properties: for B to be positive definite

☑ ① $B = B^T$

② $x \cdot Bx > 0$ for all $x \neq 0$.

① $B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$

② $x \cdot Bx = x \cdot A^T A x = x^T A^T A x = (Ax)^T Ax$

$= Ax \cdot Ax = \|Ax\|^2$

since A is invertible then $Ax \neq 0$ whenever $x \neq 0$.

Compare lu to cholesky in Julia...

```
julia> n=3
3

julia> A=rand(n,n)
3×3 Matrix{Float64}:
 0.191602   0.999168  0.690647
 0.675175   0.33967   0.607416
 0.0823803  0.660442  0.492301

julia> B=A'*A
3×3 Matrix{Float64}:
 0.499359  0.475187  0.582998
 0.475187  1.5499    1.22153
 0.582998  1.22153   1.08831
```

```
julia> L=cholesky(B).L
3×3 LowerTriangular{Float64, Matrix{Float64}}:
 0.706654   ⋅         ⋅
 0.672447  1.04772    ⋅
 0.825012  0.636386  0.0517256

julia> L*L'
3×3 Matrix{Float64}:
 0.499359  0.475187  0.582998
 0.475187  1.5499    1.22153
 0.582998  1.22153   1.08831
```

```
julia> cholesky(A)
ERROR: PosDefException: matrix is not Hermitian; Cholesky factor
ization failed.
Stacktrace:
```

← A was not even symmetric...

```
julia> C=A'+A
3×3 Matrix{Float64}:
 0.383205  1.67434   0.773027
 1.67434   0.679339  1.26786
 0.773027  1.26786   0.984603

julia> cholesky(C)
ERROR: PosDefException: matrix is not positive definite; Cholesk
y factorization failed.
Stacktrace:
```

← a way to make a symmetric matrix that is not guaranteed to be positive definite

it wasn't.

Claim: for a matrix that is positive definite finding the Cholesky factorization is (almost twice) faster than LU because of the symmetry...

```
julia> n=5000
5000

julia> A=rand(n,n);

julia> B=A'*B;
```

← test this for size 5000×5000 matrices

```
julia> @time cholesky(B);
  0.550812 seconds (5 allocations: 190.735 MiB, 11.57% gc time)

julia> @time cholesky(B);
  0.573763 seconds (5 allocations: 190.735 MiB, 7.40% gc time)

julia> @time cholesky(B);
  0.491268 seconds (5 allocations: 190.735 MiB)

julia> @time lu(B);
  0.727795 seconds (5 allocations: 190.773 MiB, 3.16% gc time)

julia> @time lu(B);
  0.715686 seconds (5 allocations: 190.773 MiB)

julia> @time lu(B);
  0.802689 seconds (5 allocations: 190.773 MiB)
```

```
julia> 0.715686/0.491268
1.4568137961357142
```

cholesky was about 45% faster when $n = 5000$.

Spectral Theorem: If $A = A^T$ then there is an orthonormal basis of eigenvectors of A.