Spectral Theorem: If $A = A^T$ then there is an orthonormal basis of eigenvectors of A.

eigenvector an eigenvalue are a pair ξ and λ such that

$$A\xi = \lambda\xi$$

converts matrix multiplication

scalar multiplication

If $A \in \mathbb{R}^{n\times n}$ and $A = A^T$ then there are n eigenvectors $\xi_1, \xi_2, \ldots \xi_n$ with eigenvalues such that

$$A\xi_i = \lambda_i \xi_i \qquad \text{and} \qquad \xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note also $\lambda_i \in \mathbb{R}$.

If also A is positive definite then $\lambda_i > 0$.

Put the vectors in a matrix: $Q = \left[\xi_1 \middle| \xi_2 \middle| \cdots \middle| \xi_n \right] \in \mathbb{R}^{n\times n}$

then $$Q^TQ = \begin{bmatrix} \xi_1^T \\ \hline \xi_2^T \\ \hline \vdots \\ \hline \xi_n^T \end{bmatrix} \left[\xi_1 \middle| \xi_2 \middle| \cdots \middle| \xi_n \right] = \left[\xi_i^T \xi_j \right]_{i,j=1\ldots n} = I$$

Since Q is square then $QQ^T = I$ and so $Q^{-1} = Q^T$.

$A\xi_i = \lambda_i \xi$ means

$$AQ = A \left[\xi_1 \middle| \xi_2 \middle| \cdots \middle| \xi_n \right] = \left[A\xi_1 \middle| A\xi_2 \middle| \cdots \middle| A\xi_n \right]$$

$$= \left[\lambda_1\xi_1 \middle| \lambda_2\xi_2 \middle| \cdots \middle| \lambda_2\xi_2 \right] = \left[\xi_1 \middle| \xi_2 \middle| \cdots \middle| \xi_n \right] \overbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}^{D}$$

Thus $AQ = QD$ and $\underbrace{AQQ^T}_{\text{identity}} = QDQ^T$

Then $A = QDQ^T$.

That's it for now, but we'll return to the spectral theorem in the next chapter...

Solving equations: like find and x such that $f(x)=0$.

Bisection, Secant Method, false position, Newton's method

Get an approximation x to the exact solution p.

$f(x) \approx 0$ ↗ approx solution

$f(p) = 0$ ↑ exact solution

Question: If I calculate how close $f(x)$ is to 0 what does that say about how close x is to p?

$$\text{Error} = |x-p|$$

Calculus Taylor's theorem

$$f(x) = \underbrace{f(p)}_{0} + (x-p)f'(p) + R_2$$

$R_2 = \frac{(x-p)^2}{2} f''(\xi)$ for some ξ between x and p.

$$f(x) \approx (x-p)f'(p)$$

$$x-p \approx \frac{f(x)}{f'(p)}$$

Alternatively:

$$0 = f(p) = f(x) + (p-x)f'(x) + R_2$$

$R_2 = \frac{(p-x)^2}{2} f''(\xi)$ for some ξ between x and p.

$$f(x) + (p-x)f'(x) \approx 0$$

$$x-p \approx \frac{f(x)}{f'(x)}$$

Thus

$$|x-p| \approx \left|\frac{f(x)}{f'(x)}\right|$$

convert the error in how close $f(x)$ is to zero

to the error how close x is to p.

Backwards error estimation: the error in the output $f(x)$ ← residual is used to find the error in the input x.

Do Backwards error estimation for matrix equation $Ax=b$.

Suppose x is the exact solution and $\hat{x}$ is the approximation.

Then $A\hat{x} \approx b$ and $Ax = b$ ← $x = A^{-1}b$

approximation   exact

$A\hat{x} = b + r$ $\hat{x} = A^{-1}(b+r)$

Residual error $r = A\hat{x} - b$ goal to convert r back to say something about the error in $\hat{x}$.

$$\hat{x} - x = \cancel{A^{-1}b} - \cancel{A^{-1}}(b+r) = -A^{-1}r$$

$$\|\hat{x}-x\| = \|A^{-1}r\| \le \|A^{-1}\|\,\|r\|$$

distance between two vectors

size of residual erro

size of the inverse matrix.

(need an estimate on the size of the inverse matrix)

For example

$$\|\hat{x}-x\| = \sqrt{\sum (\hat{x}_i - x_i)^2}$$

① How to compute matrix norms and what do they mean... Chapter 2.5.

② What about relative errors?

relative error $= \dfrac{\|\hat{x}-x\|}{\|x\|}$