# Vector norms and Matrix Norms

Euclidean distance from the origin $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for $x \in \mathbb{R}^n$

distance between two points (points in $\mathbb{R}^n$) $\|x-y\|$ for $x, y \in \mathbb{R}^n$

Dot product $x \in \mathbb{R}^n$ $y \in \mathbb{R}^n$ then $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

Note: I can write the norm in terms of the dot product.

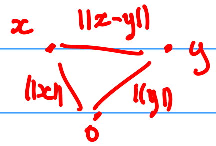
$$\|x\| = \sqrt{x \cdot x}$$

Question: Can you write the dot product in term of the norm?

$x \cdot y =$ "Something with norms in it but no dot products here or $\cos\theta$"

— polarization identity —

## Properties of a Vector norm:




1. $\|x\| = 0$ if only if $x = 0$
2. $\|x \pm y\| \leq \|x\| + \|y\|$ triangle inequality
3. $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$

consequence of the ①–③ for a vector norm is
$|x \cdot y| \leq \|x\| \|y\|$
look a little like ④ for matrices but it's different...

## Properties of a Matrix norm:

1. $\|A\| = 0$ if only if $A = 0$
2. $\|A+B\| \leq \|A\| + \|B\|$ triangle inequality
3. $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in \mathbb{R}$
4. $\|AB\| \leq \|A\| \|B\|$

$A = x^T$ $B = y$

$AB = x^T y = x \cdot y$

Could treat the matrix as a big vector written is some sort of table

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij})^2}$$

satisfies the properties of a matrix norm...

This ignores the fact that a matrix represents a linear function

$f(x) = Ax$ ← the linear function corresponding to A...

$g(x) = Bx$

$f \circ g = AB$ ← matrix multiplication

Better to use something ~~less artificial~~... more complicated

$$\underset{\text{matrix}}{\|A\|_2} = \max\{\underset{\text{vector norm}}{\|Ax\|} : \underset{\text{vector norm}}{\|x\|} \leq 1\} = \max\{\|Ax\| : \|x\| = 1\}$$

also satisfies the properties of a norm: $\|A\|_2 \leq \|A\|_F$

induced matrix norm by the vector norm is the smallest norm such that $\|Ax\| \leq \|A\| \|x\|$ for all $x \in \mathbb{R}^n$.

these come from the vector norms

① $\|A\| = 0$ if only if $A = 0$ — $x^T$ it's

② $\|A+B\| \leq \|A\| + \|B\|$ triangle inequality

③ $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in \mathbb{R}$

we'll check this ④ $\|AB\| \leq \|A\| \|B\|$ — $A = x^T$

$$\|AB\| = \max\{\|A\underbrace{Bx}_{y=Bx}\| : \|x\|=1\}$$

$$= \max\{\|Ay\| : y=Bx \text{ and } \|x\|=1\}$$

y=0 is the min value... doesn't change the maximum.

$$= \max\left\{\left\|A\frac{y}{\|y\|}\right\|\|y\| : y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\right\}$$

maximum of a product is less or equal the product of maxima

$$\leq \max\left\{\left\|A\frac{y}{\|y\|}\right\| : y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\right\}$$

$$\times \max\{\|y\| : y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\} \leq \|A\|\|B\|$$

Where

substitute back in.

$$\max\{\|y\| : y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\}$$

$$= \max\{\|Bx\| : \|x\|=1\} = \|B\|$$

and

$$\max\left\{\left\|A\frac{y}{\|y\|}\right\| : y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\right\}$$

$w=\frac{y}{\|y\|}$

$$= \max\left\{\|Aw\| : w=\frac{y}{\|y\|} \text{ and } y\neq 0 \text{ and } y=Bx \text{ and } \|x\|=1\right\}$$

remove these conditions — max gets bigger

$$\leq \max\{\|Aw\| : \|w\|=1\} = \|A\|$$

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To compute the 'induced matrix norm', we'll use the spectral theorem. Please review that for next time.