To compute the 'induced matrix norm', we'll use the spectral theorem. Please review that for next time.

$$\|A\| = \max\{\|Ax\| : \|x\|=1\}$$

Spectral theorem: If $B = B^T$ then B has an orthonormal basis of eigenvectors.

$$B\xi_i = \lambda_i \xi_i \qquad \xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Need a symmetric matrix to apply spectral theorem

$$B = A^T A$$

that's symmetric.

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B.$$

also positive definite...

How to use B to compute $\|A\|$?

$$\|A\| = \max\{\underbrace{\|Ax\|}_{\text{need to maximize this.}} : \|x\|=1\} = \sqrt{\max\{\|Ax\|^2 : \|x\|=1\}}$$

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T Ax = x^T A^T A x = x^T B x$$

where $B = A^T A$

Idea: write x in terms of the basis of orthonormal eigenvectors of B.

note n eigenvectors if A was $m \times n$.

$$Q = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix}$$

$$BQ = Q \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{D}$$

Recall: $Q^TQ = I$ and $QQ^T = I$, that is Q is an orthogonal matrix

want to write

$$x = c_1\xi_1 + c_2\xi_2 + \cdots + c_n\xi_n .$$

$$x = Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Qc \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$Q^Tx = \cancel{Q^TQ}c = c$$

$c = Q^Tx$, $\quad x = Qc \quad$ make this invertible change of vbls

Therefore, continuing

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^TAx = x^TA^TAx = x^TBx$$

$$= (Qc)^TBQc = c^TQ^TBQ\,c$$

Recall

$$BQ = Q\underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{D}$$

$BQ = QD$

(columns of Q are eigenvectors of B.)

$$\|Ax\|^2 = c^T Q^T Q D c = c^T D c$$

One more thing... need to know $\|c\|$ ... Why?

Change of variables $x = Qc$, $c = Q^T x$

$$\|A\| = \sqrt{\max\{\|Ax\|^2 : \|x\| = 1\}} = \sqrt{\max\{c^T D c : \|Qc\| = 1\}}$$

$$\|Qc\|^2 = (Qc)^T Qc = c^T \cancel{Q^T} Qc = c^T c = \|c\|^2$$

$\sum c_i^2$

Therefore

$$\|A\| = \sqrt{\max\{c^T D c : \|c\| = 1\}}$$

since D is diagonal finding this maximum is easier...

$$c^T D c = c \cdot Dc = c \cdot \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} c = \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\|A\| = \sqrt{\max\left\{\sum_{i=1}^{n} \lambda_i c_i^2 : \sum c_i^2 = 1\right\}}$$

Since $\sum c_i^2 = 1$ then

$\sum_{i=1}^{n} \lambda_i c_i^2$ is a weighted average of the λ_i's.

it must be some where between the smallest and largest λ_i.

If λ_q is the largest λ, then setting $c_q = 1$ and $c_i = 0$ for $i \neq q$

Gives that $\sum_{i=1}^{n} \lambda_i c_i^2 = \lambda_q = \max\{\lambda_i : i = 1, \ldots, n\}$

$$\|A\| = \sqrt{\max\left\{\sum_{i=1}^{n} \lambda_i c_i^2 : \sum c_i^2 = 1\right\}}$$

singular values of A.

$$= \sqrt{\max\{\lambda_i : i = 1, \ldots, n\}} = \max\{\sqrt{\lambda_i} : i = 1, \ldots, n\}$$

eigenvalues of $B = A^T A$ ....

$$\|A\|_s = \sqrt{\rho(A^T A)},$$

where $\rho(M)$ is the **spectral radius** of M, that is, the largest eigenvalue of M in absolute value. The square roots of the eigenvalues of $A^T A$ (which must be nonnegative) are called the **singular values** of A so we can also say that $\|A\|_s$ is equal to the largest singular value of A.