

```

julia> A=[1 2 3; 4 5 6; 1 0 1]
3x3 Matrix{Int64}:
 1  2  3
 4  5  6
 1  0  1

julia> using LinearAlgebra

julia> opnorm(A)
9.575782151104967

```

$\leftarrow \max \{ \sqrt{\lambda_i} : \lambda_i \text{ is an eigenvalue of } B=A^T A \}$

```

julia> B=A'*A
3x3 Matrix{Int64}:
 18  22  28
 22  29  36
 28  36  46

julia> eigvals(B)
3-element Vector{Float64}:
 0.4712035626512613
 0.8331926319282537
 91.69560380542052

```

```

julia> sqrt(91.69560380542052)
9.575782151104969

```

same

$\|A\|_2$
 λ means
 Spectrum

\leftarrow eigenvalues of $B=A^T A$

$\sqrt{\lambda_i}$ for the
 largest λ_i

What about $\|A\|_F = \sqrt{\sum |a_{ij}|^2}$

$\|A\|_2 \leq \|A\|_F$

```

julia> A.*A
3x3 Matrix{Int64}:
 1  4  9
 16 25 36
 1  0  1

julia> sum(A.*A)
93

julia> sqrt(93)
9.643650760992955

```

```

julia> norm(A) = \|A\|_F
9.643650760992955

```

is bigger

same

$\|A\|_2 \leq \|A\|_F$

Supposed to be that $\|Ax\|_2 \leq \|A\|_2 \|x\|_2 \leq \|A\|_F \|x\|_2$

Check this...

$\|b\|$ compare $\|A\| \|x\|$

```
julia> x=[1,2,3]
3-element Vector{Int64}:
 1
 2
 3

julia> b=A*x
3-element Vector{Int64}:
14
32
 4
```

```
Smaller julia> norm(b)
35.156791662493895 = ||Ax||

bigger julia> opnorm(A)*norm(x)
35.829296019819964 = ||A||_2||x||
```

Backwards error analysis for the problem $Ax=b$.

- ① Find an approximation \hat{x} in some complicated way where it's difficult to track the errors.
- ② Determine the error in \hat{x} by plugging it in.

☑ ① How to compute matrix norms and what do they mean... Chapter 2.5.

☐ ② What about relative errors?

$$\text{relative error} = \frac{\|\hat{x} - x\|}{\|x\|}$$

Let x be the exact solution so $Ax=b$
 \hat{x} be an approximation with residual $r = A\hat{x} - b$

$$\hat{x} - x = A^{-1}b - A^{-1}(b+r) = -A^{-1}r$$

$$\|\hat{x} - x\| = \|A^{-1}r\| \leq \|A^{-1}\| \|r\|$$

For the relative error, divide by $\|x\|$

$$\frac{\|\hat{x} - x\|}{\|x\|} = \frac{\|A^{-1}r\|}{\|x\|} \leq \frac{\|A^{-1}\| \|r\|}{\|x\|}$$

Use matrix norms to bound $\frac{\|r\|}{\|x\|}$ here

$$Ax = b$$

$$\|b\| = \|Ax\| \leq \|A\| \|x\|$$

Therefore

$$\frac{\|x\|}{\|b\|} \leq \frac{\|A\|}{\|b\|}$$

condition number of the matrix A

Thus

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \frac{\|A^{-1}\| \|r\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|r\|}{\|b\|}$$

Define $\kappa(A) = \|A\| \|A^{-1}\|$ condition number..

```

julia> A
3x3 Matrix{Int64}:
 1  2  3
 4  5  6
 1  0  1

julia> cond(A)
13.949864032758871
    
```

builtin function to compute the condition number..

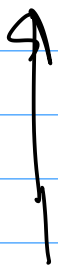
Note, Julia did Not find A^{-1} and then compute $\|A^{-1}\|$ in order to find the condition number-- There are ways to find this directly...

However, we can verify that $\kappa(A) \approx 13.949...$ by finding $\|A^{-1}\|$ ourselves.

```
julia> Ainv=inv(A)
3x3 Matrix{Float64}:
-0.833333  0.333333  0.5
-0.333333  0.333333 -1.0
 0.833333 -0.333333  0.5

julia> opnorm(Ainv)
1.4567858596437637

julia> kappaA=opnorm(A)*opnorm(Ainv)
13.949864032758859
```



The rounding is different means Julia used a different method to find the condition number...

This is not a very big number so

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq 13,949 \frac{\|r\|}{\|b\|}$$

```

julia> n=100
100

julia> x=ones(n);

julia> A=rand(n,n);

julia> b=A*x;

julia> xapp=A\b;

julia> norm(xapp-x)
1.2293458343110473e-12

```

← absolute error

```

julia> r=A*xapp-b;

julia> cond(A)*norm(r)/norm(b)
2.415860514820197e-12

```

← estimate of relative error

```

julia> norm(xapp-x)/norm(x)
1.2293458343110474e-13

```

← actual relative error

```

julia> cond(A)
7077.094969786443

```

← The power of 10's in the condition number indicates how many digits in it could be expected

For example, if the residual error is computed using double-precision arithmetic, then $\|r\|/\|b\|=0$ means only to 15 significant digits. Since these are relative errors, that means plus or minus 0.5×10^{-15} . This multiplied by the condition number implies the computer can not tell the difference between approximations for x that agree to within $15-3=12$ significant digits.