```
julia> A=[1 2 3; 4 5 6; 1 0 1]
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 1  0  1

julia> using LinearAlgebra

julia> opnorm(A)
9.575782151104967
```

$\max\{\sqrt{\lambda_i} : \lambda_i$ is an eigenvalue of $B = A^TA\}$

same

```
julia> B=A'*A
3×3 Matrix{Int64}:
 18  22  28
 22  29  36
 28  36  46

julia> eigvals(B)
3-element Vector{Float64}:
  0.4712035626512613
  0.8331926319282537
 91.69560380542052
```

eigenvalues of $B = A^TA$

```
julia> sqrt(91.69560380542052)
9.575782151104969
```

$\|A\|_S$

S means Spectrum

$\sqrt{\lambda_i}$ for the largest λ_i

$\|A\|_S \leq \|A\|_F$

What about $\|A\|_F = \sqrt{\sum |a_{ij}|^2}$

bigger

```
julia> norm(A)
9.643650760992955
```

$= \|A\|_F$

```
julia> A.*A
3×3 Matrix{Int64}:
  1   4   9
 16  25  36
  1   0   1

julia> sum(A.*A)
93

julia> sqrt(93)
9.643650760992955
```

Same

$\|A\|_S \leq \|A\|_F$

Supposed to be that $\|\underbrace{Ax}_{b}\|_2 \leq \|A\|_S \|x\|_2 \leq \|A\|_F \|x\|_2$

Check this...

$\|b\|$ compare $\|A\| \|x\|$

```
julia> x=[1,2,3]
3-element Vector{Int64}:
 1
 2
 3

julia> b=A*x
3-element Vector{Int64}:
 14
 32
  4
```

```
julia> norm(b)
35.15679162493895

julia> opnorm(A)*norm(x)
35.829296019819964
```

smaller $35.15679162493895 = \|Ax\|$

bigger $35.829296019819964 = \|A\|\|x\|$

Backwards error analysis for the problem $Ax=b$.

1. Find an approximation $\hat{x}$ is some complicated way where it's difficult to track the errors.
2. Determine the error in $\hat{x}$ by plugging it in.

1. How to compute matrix norms and what do they mean... Chapter 2.5.
2. What about relative errors?

relative error $= \dfrac{\|\hat{x}-x\|}{\|x\|}$

Let x be the exact solution so $Ax=b$
$\hat{x}$ be an approximation with residual $r = A\hat{x} - b$

$$\hat{x} - x = \cancel{A^{-1}b} - \cancel{A^{-1}}(b+r) = -A^{-1}r$$

$$\|\hat{x} - x\| = \|A^{-1}r\| \leq \|A^{-1}\|\,\|r\|$$

For the relative error, divide by $\|x\|$

$$\frac{\|\hat{x} - x\|}{\|x\|} = \frac{\|A^{-1}r\|}{\|x\|} \leq \frac{\|A^{-1}\|\,\|r\|}{\|x\|}$$

Use matrix norms to bound $\frac{1}{\|x\|}$ here

$$Ax = b$$

$$\|b\| = \|Ax\| \leq \|A\|\,\|x\|$$

Therefore

$$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

condition number of the matrix A

Thus $$\frac{\|\hat{x} - x\|}{\|x\|} \leq \frac{\|A^{-1}\|\,\|r\|}{\|x\|} \leq \|A\|\,\|A^{-1}\|\frac{\|r\|}{\|b\|}$$

Define $\kappa(A) = \|A\|\,\|A^{-1}\|$ condition number...

```
julia> A
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 1  0  1

julia> cond(A)
13.949864032758871
```

builtin function to compute the condition number...

Note, Julia did Not find A^{-1} and then compute $\|A^{-1}\|$ in order to find the condition number... There are ways to find this directly...

However, we can verify that $\kappa(A) \approx 13.949\ldots$ by finding $\|A^{-1}\|$ ourselves.

```
julia> Ainv=inv(A)
3×3 Matrix{Float64}:
 -0.833333   0.333333   0.5
 -0.333333   0.333333  -1.0
  0.833333  -0.333333   0.5

julia> opnorm(Ainv)
1.4567858596437637

julia> kappaA=opnorm(A)*opnorm(Ainv)
13.949864032758859
```

The rounding is different means Julia used a different method to find the condition number...

This is not a very big number so

$$\frac{\|\hat{x}-x\|}{\|x\|} \lesssim 13.949 \frac{\|r\|}{\|b\|}$$

```
julia> n=100
100

julia> x=ones(n);

julia> A=rand(n,n);

julia> b=A*x;

julia> xapp=A\b;

julia> norm(xapp-x)
1.229345834311O473e-12
```

absolute error

```
julia> r=A*xapp-b;

julia> cond(A)*norm(r)/norm(b)
2.415860514820197e-12
```

estimate of relative error

```
julia> norm(xapp-x)/norm(x)
1.229345834311O474e-13
```

actual relative error...

```
julia> cond(A)
7077.094969786443
```

The power of 10's in the condition number indicates how many digits in $\hat{x}$ could be expected

For example,
if the residual error is computed using double-precision arithmetic, then ||r||/||b||=0 means only to 15 significant digits. Since these are relative errors, that means plus or minus 0.5 x 10^(-15). This multiplied by the condition number implies the computer can not tell the difference between approximations for x that agree to within 15-3=12 significant digits.