The QR decomposition of a matrix

$A \in \mathbb{R}^{m \times n}$ then $A = QR$ where $Q^T Q = I$

and R is upper triangular.

The algorithm to find QR in and introductory linear algebra class is call Gram-Schmidt.

$$\underset{m\times n}{A} = \underset{m\times ?}{Q}\underset{?\times n}{R}$$

these two question marks have to be the same number... but what?

Two kinds of QR factorizations

1. Full QR factorization. {Q is square, R is not}
2. Reduced QR factorization. {R is square, Q is not} ← The reduced comes from Gram-Schmidt naturally.

What's the difference?. The dimensions of Q and R...

Reduced QR decomposition.

$A \in \mathbb{R}^{m \times n}$ $m \geq n$

well determined $m=n$
over determined $m>n$.

$$\underset{m\times n}{A} = \underset{m\times n}{Q}\ \underset{n\times n}{R}$$

R is square...

Gram-schmidt Algorithm.

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \quad \text{where} \quad u_j \in \mathbb{R}^m \quad \text{for } j=1,\dots,n$$

Use column operations to factor A:

Case $n=3$

$$\begin{cases} v_1 = u_1 \\ v_2 = u_2 - (q_1 \cdot u_2) q_1 \\ v_3 = u_3 - (q_1 \cdot u_3) q_1 - (q_2 \cdot u_3) q_2 \end{cases}$$

$$u_2 = v_2 + (q_1 \cdot u_2) q_1 = \boxed{\|v_2\| q_2 + (q_1 \cdot u_2) q_1}$$

$$u_3 = v_3 + (q_1 \cdot u_3) q_1 + (q_2 \cdot u_3) q_2$$
$$= \boxed{\|v_3\| q_3 + (q_1 \cdot u_3) q_1 + (q_2 \cdot u_3) q_2}$$

$$q_1 = \frac{v_1}{\|v_1\|}$$

$$q_2 = \frac{v_2}{\|v_2\|} \qquad v_2 = \|v_2\| q_2$$

$$q_3 = \frac{v_3}{\|v_3\|} \qquad v_3 = \|v_3\| q_3$$

$$Q = \left[q_1 \middle| q_2 \middle| q_3 \right] \qquad R = \begin{bmatrix} \|v_1\| & q_1 \cdot u_2 & q_1 \cdot u_3 \\ 0 & \|v_2\| & q_2 \cdot u_3 \\ 0 & 0 & \|v_3\| \end{bmatrix}$$

Note this algorithm focuses on R. why... we do column operations and R 'stores' them.

$$QR = \left[\|v_1\| q_1 \;\middle|\; \boxed{(q_1 \cdot u_2) q_1 + \|v_2\| q_2} \;\middle|\; \boxed{(q_1 \cdot u_3) q_1 + (q_2 \cdot u_3) q_2 + \|v_3\| q_3} \right]$$

$$= \left[u_1 \middle| u_2 \middle| u_3 \right] = A$$

For arbitrary n

$$v_1 = u_1 \qquad q_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = u_2 - (q_1 \cdot u_2) q_1 \qquad q_2 = \frac{v_2}{\|v_2\|}$$

$$v_3 = u_3 - (q_1 \cdot u_3) q_1 - (q_2 \cdot u_3) q_2 \qquad q_3 = \frac{v_3}{\|v_3\|}$$

$$v_k = u_k - \sum_{l=1}^{k-1} (q_l \cdot u_k) q_l \qquad q_k = \frac{v_k}{\|v_k\|}$$

for $k = 4, 5, \ldots, n$.

The Gram-Schmidt focuses on making R and gets Q as a result:

Is this $\boxed{Q^T Q = I}$ true?

rounding errors mess this orthogonality up..

Householder focuses on ensuring $Q^T Q = I$ and gets R as a result. Interestingly there are choices that can be made that will further reduce rounding error when using Householder.