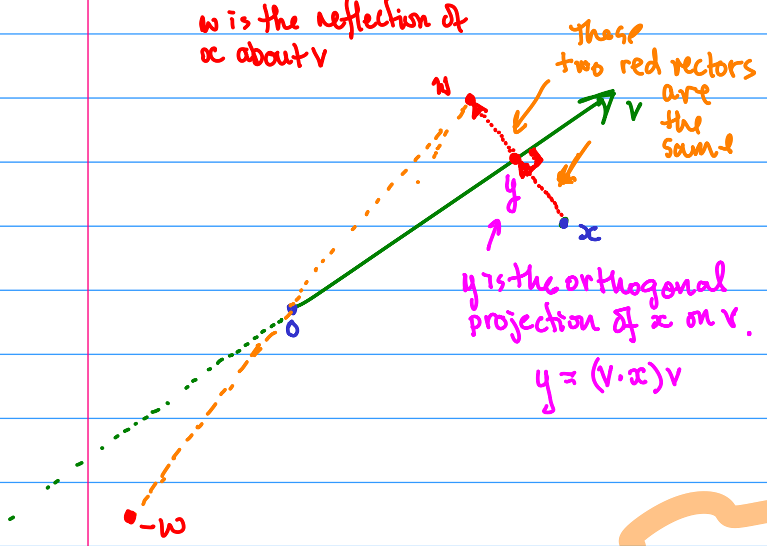
Householder Reflectors

Let v be a unit vector.



$y = (v \cdot x)v$

Goal find w...

$w = y + (w - y)$

$= y + (y - x)$

$= 2y - x$

$= 2(v \cdot x)v - x$ factor the x out

$= 2(v^T x)v - x$ just a scalar

$= 2v(v^T x) - x$

$= (2vv^T - I)x$ reflection that maps x to w

$= -Hx$

Householder reflector

Let $H = I - 2vv^T$

Thus $Hx = -w$

Since H is a reflection then its an orthogonal matrix

Verify that H is orthogonal: $\boxed{\checkmark}$ ① H is square

② $H^T H = I$

actually ① + ② also mean $H^T = H^{-1}$.

This part shows that $H^T = H$ or that H is symmetric.

$$H^T H = (I - 2vv^T)^T (I - 2vv^T)$$

$$= (I^T - (2vv^T)^T)(I - 2vv^T)$$

$$= (I - 2v^{TT}v^T)(I - 2vv^T)$$

$$= (I - 2vv^T)(I - 2vv^T) = I - 2vv^T - 2vv^T + 4vv^Tvv^T$$

recall v is a unit vector so $v \cdot v = v^T v = 1$

$$= 1 - 2vv^T - 2vv^T + 4vv^T = I$$

Not a surprise because all reflections are orthogonal.

- Note that rotations are also orthogonal and a similar way to make the QR factorization uses rotatations instead of reflections.

Given a matrix $A \in \mathbb{R}^{m \times n}$ ← columns

Idea find v a unit vector so that $H = I - 2vv^T$

$$HA = H\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} Ha_1 & Ha_2 & \cdots & Ha_n \end{bmatrix} = \begin{bmatrix} c & Ha_2 & \cdots & Ha_n \\ 0 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

Just solve for v: $Ha_1 = ce_1$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

unit vector

$$\boxed{(I - 2vv^T)a_1 = ce_1}$$ ← solve for v.

Three vectors... dot the equation into these vectors to get more equations...

First v:

$$v^T(I - 2vv^T)a_1 = v^T c e_1$$

$$v \cdot e_1 = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_1$$

$$v^T a_1 - 2v^Tv v^T a_1 = c v^T e_1$$

$$v^T a_1 - 2v^T a_1 = c v_1$$

$$\boxed{-v^T a_1 = c v_1}$$

Next a_1:

$$a_1^T(I - 2vv^T)a_1 = a_1^T c e_1$$

$$a_1^T a_1 - 2a_1^T v v^T a_1 = c a_{11}$$ ← first entry of first column of A.

$$\|a_1\|^2 - 2(a_1 \cdot v)(v \cdot a_1) = c a_{11}$$

$$\boxed{\|a_1\|^2 - 2(v^T a_1)^2 = c a_{11}}$$

$$\|a_1\|^2 - 2(c v_1)^2 = c a_{11}$$

Finally e_1

$$e_1^T(I - 2vv^T)a_1 = e_1^T c e_1$$

$$e_1^T a_1 - 2e_1^T v v^T a_1 = c$$

$$\boxed{a_{11} - 2v_1 v^T a_1 = c}$$

$$c a_{11} - 2c v_1 v^T a_1 = c^2$$

$$c a_{11} + 2c v_1 c v_1 = c^2$$

$$\|a_1\|^2 - \cancel{2(cv_1)^2} = \cancel{ca_{11}}$$

$$\cancel{ca_{11}} + \cancel{2cv_1 cv_1} = c^2$$

$$\|a_1\|^2 = c^2$$

So I know $c = \pm\|a_1\|$

what is v

$\|a_1\|^2 = c^2$ $\qquad\qquad$ $(I - 2vv^T)a_1 = ce_1$

Now solve for v: $\qquad a_1 - 2v(v^Ta_1) = ce_1$

$$v = \frac{a_1 - ce_1}{2v^Ta_1} = \frac{a_1 - ce_1}{\|a_1 - ce_1\|}$$

since its a unit vector...

choose $c = \pm\|a_1\|$ so the term in the norm is a large as possible...