Power Method

Algorithm 27.1. Power Iteration

$v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$
for $k = 1, 2, \ldots$
    $w = Av^{(k-1)}$ — apply A
    $v^{(k)} = w/\|w\|$ — normalize — may not converge...
    $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ — Rayleigh quotient — these are converging...

$$v_k = \frac{c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|}$$

where ξ_i were eigenvectors of A with eigenvalues λ_i.

and $v_0 = c_1\xi_1 + c_2\xi_2 + \cdots + c_n\xi_n$

Since there are all sorts of eigenvectors of different multiples of each other corresponding to λ_1 then it's not clear which of those eigenvectors

$$\lim_{k\to\infty} v_k$$

should have converged to in the first place...

$$\lambda^{(k)} = v_k \cdot A v_k$$

$$= \frac{c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|} \cdot A \frac{c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|}$$

$$= \frac{\sum_{i=1}^{m} c_i\lambda_i^k\xi_i}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|} \cdot \frac{\sum_{j=1}^{n} c_j\lambda_j^k A\xi_j}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|}$$

use the fact that $A\xi_i = \lambda_i\xi_i$

multiply everything together...

$$= \frac{\boxed{\sum_{i=1}^{n} c_i \lambda_i^k \xi_i}}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|} \cdot \frac{\sum_{j=1}^{n} c_j \lambda_j^{k+1} \xi_j}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2 + \cdots + c_n\lambda_n^k\xi_n\|}$$

$$= \frac{\sum_{i,j} c_i\lambda_i^k \xi_i \cdot c_j \lambda_j^{k+1}\xi_j}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k + \cdots + c_n\lambda_n^k\xi_n\|^2} = \frac{\sum_{i,j} c_i c_j \lambda_i^k \lambda_j^{k+1} \xi_i \cdot \xi_j}{\|c_1\lambda_1^k\xi_1 + c_2\lambda_2^k + \cdots + c_n\lambda_n^k\xi_n\|^2}$$

$$= \frac{\sum_{i,j} c_i c_j \boxed{\frac{\lambda_i^k \lambda_j^{k+1}}{\lambda_1^{2k}}} \xi_i \cdot \xi_j}{\|c_1\xi_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k \xi_n\|^2}$$

Now take limit as $k \to \infty$

$$\lim_{k\to\infty} \frac{\lambda_i^k \lambda_j^{k+1}}{\lambda_1^{2k}} = \lim_{k\to\infty} \left(\frac{\lambda_i}{\lambda_1}\right)^k \left(\frac{\lambda_j}{\lambda_1}\right)^k \lambda_j = \begin{cases} 0 & \text{if } i \neq 1 \text{ or } j \neq 1 \\ \lambda_1 & \text{if } i = 1 \text{ and } j = 1 \end{cases}$$

Therefore

$$\lim_{k\to\infty} \frac{\sum_{i,j} c_i c_j \boxed{\frac{\lambda_i^k \lambda_j^{k+1}}{\lambda_1^{2k}}} \xi_i \cdot \xi_j}{\|c_1\xi_1 + c_2\underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^k}_{\to 0} + \cdots + c_n\underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^k}_{\to 0} \xi_n\|^2} = \frac{c_1^2 \lambda_1 \|\xi_1\|^2}{\|c_1\xi_1\|^2} = \lambda_1$$

This shows that $\lambda^{(k)}$ in the power method converge to the eigenvalue of largest magnitude...

How to find the other eigenvalues?
How to find the eigenvector?

Note even though the v_k don't converge in general, they get closer to an eigenvector in the limit...

$$A v_k \approx \lambda^k v_k$$

when k is large. So one idea is to wait until the $\lambda^{(k)}$'s converge and then take the v_k as the corresponding eigenvector.

Another idea after finding the eigenvalue, solve for the $\mathrm{Nul}(A-\lambda I)$ to obtain the eigenvector.

means solve $(A-\lambda I)x = 0$ for $x \neq 0$.

sort of like inverting $A-\lambda I$, except that it's not invertible...

Think about the inverse of $A-\mu I$ where μ is any value....

If μ is not an eigenvalue of A then $A-\mu I$ is invertible

$$B = (A-\mu I)^{-1}$$

Question, what are the eigenvalues of B?

Answer: use the spectral mapping theorem.

Suppose $A\xi = \lambda\xi$.

$$B\xi = (A-\mu I)^{-1}\xi$$

$$(A-\mu I)B\xi = \xi$$

Try simpler idea $C = A-\mu I$ what are the eigenvalues of C?

Answer: use the spectral mapping theorem.

Suppose $A\xi = \lambda\xi$.

$$C\xi = (A-\mu I)\xi = A\xi - \mu\xi = \lambda\xi - \mu\xi = (\lambda-\mu)\xi$$

Conclusion: the eigenvalues of C are the same as the eigenvalues of A, except shifted by μ.

Now return to ask what are the eigenvalues of B.

Since $B = (A-\mu I)^{-1} = C^{-1}$

Since $C\xi = (\lambda-\mu)\xi$

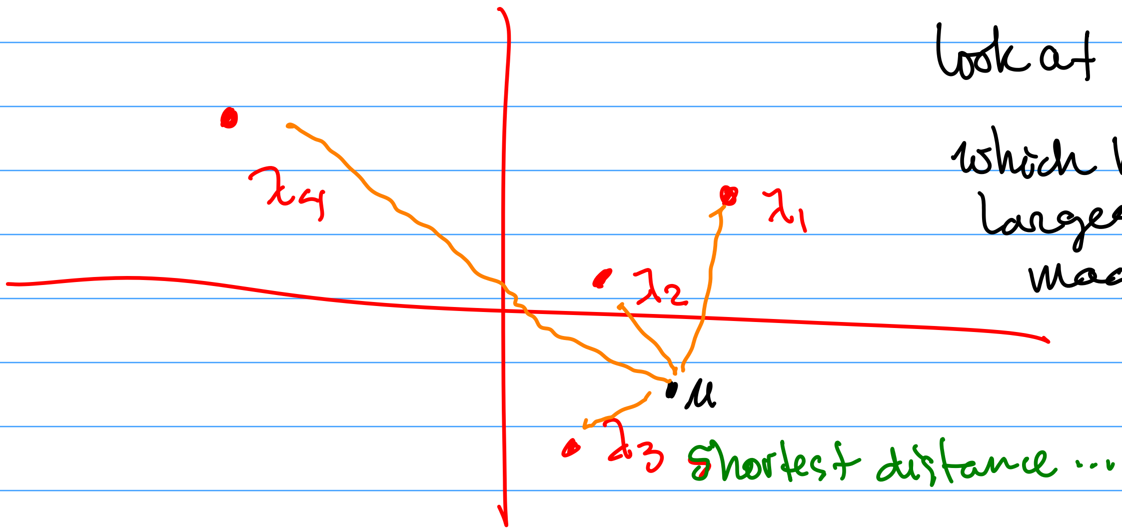
$$C^{-1}C\xi = C^{-1}(\lambda-\mu)\xi$$

$$\xi = (\lambda-\mu)C^{-1}\xi$$

Therefore $\quad C^{-1}\xi = \dfrac{1}{\lambda - \mu}\xi$

So the eigenvalues of B are $\dfrac{1}{\lambda-\mu}$ where λ are the eigenvalues of A.

What's the use



look at $\dfrac{1}{\lambda_i - \mu}$ which has the largest magnitude

Apply the power method to $B = (A-\mu I)^{-1}$ then it will converge to the eigenvalue

$$\frac{1}{\lambda_3 - \mu} \quad \text{of } B \ldots$$

Then transform this back to find λ_3 the eigenvalue of A.

Finals week begins | Thursday, 12/15/2022

Friday 12/16

Friday, Second day of finals

Class Time	Class Day(s)	Final Meeting Time
10 a.m.	Monday/Wednesday/Friday (MWF)	9:50-11:50 a.m.
Noon	Monday/Wednesday/Friday (MWF)	12:10-2:10 p.m.
1 p.m.	Friday (F)	7:30-9:30 a.m.
2:30 p.m.	Friday (F)	7:10-9:10 p.m.

Final exam of Friday of finals week.