Algorithm 27.3. Rayleigh Quotient Iteration

$v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$
$\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$ = corresponding Rayleigh quotient
for $k = 1, 2, \ldots$
  Solve $(A - \lambda^{(k-1)} I) w = v^{(k-1)}$ for w apply $(A - \lambda^{(k-1)} I)^{-1}$
  $v^{(k)} = w / \|w\|$ normalize
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ Rayleigh quotient

provides an approximation of the eigenvalue...

approx of an eigenvector

Cubically convergent...

$$|\lambda^{(k+1)} - \lambda_J| = O(|\lambda^{(k)} - \lambda_J|^3)$$

approximations

exact eigenvalue

Claim: the Rayleigh quotient is the least squares approximation of λ.

$$Av = \lambda v$$

approximate eigenvector

$$v\lambda = Av$$

Think of this as an overdetermined system of linear equations where v is known and λ is the parameter...

In the language of a linear algebra class

$A = v$

$x = \lambda$

$b = Av$

assume $\|v\| = 1$

$$\lambda = (v^T v)^{-1} v^T A v$$

$$\lambda = (\|v\|^2)^{-1} v^T A v$$

$$\lambda = \frac{v^T A v}{\|v\|^2} = v^T A v$$

least squares for $Ax = b$.

Thus $A^T A x = A^T b$

$x = (A^T A)^{-1} A^T b$

Thus λ is the best fit in the least squares sense for the eigenvalue of v.

$$\lambda \approx v^T A v.$$

Lets suppose ξ is the exact eigenvector...

and $\|\xi\| = 1$.

Then $\lambda = \xi^T A \xi$

exact equality...

Suppose $\|v - \xi\| < \varepsilon$. Estimate $|\lambda - v^T A v|$

$$\lambda - v^T A v = \xi^T A \xi - v^T A v$$

Comparing two things with two thing different (the first and the last vector are different) introduce an intermediate point of comparison.

$$= \xi^T A \xi - \xi^T A v + \xi^T A v - v^T A v$$

Note only one term is different in each of the pairs...

eigenvector is nice

$$= \xi^T A(\xi - v) + (\xi - v)^T A v - (\xi - v)^T A \xi + (\xi - v)^T A \xi$$

Assume $A = A^T$

$$= \xi^T A(\xi - v) + (\xi - v)^T A(v - \xi) + (\xi - v)^T A\xi$$

$$= \xi \cdot A(\xi - v) + (\xi - v)\cdot A(v - \xi) + (\xi - v)\cdot A\xi$$

$$= A^T\xi \cdot (\xi - v) + (\xi - v)\cdot A(v - \xi) + (\xi - v)\cdot A\xi$$

$$= \boxed{2(\xi - v)\cdot A\xi} + \boxed{(\xi - v)\cdot A(v - \xi)}$$

The assumption $\|v - \xi\| < \varepsilon$ means...

$$\|v - \xi\|^2 \leq \varepsilon^2 \qquad (v - \xi)\cdot(v - \xi) \leq \varepsilon^2$$

$v \cdot v = \|v\|^2 = 1$

$\xi \cdot \xi = \|\xi\|^2 = 1$

$$v \cdot v - v \cdot \xi - \xi \cdot v + \xi \cdot \xi \leq \varepsilon^2$$

$$1 - 2 v \cdot \xi + 1 \leq \varepsilon^2$$

$$2(1 - v \cdot \xi) \leq \varepsilon^2$$

Now since $A\xi = \lambda \xi$ then

$\|\xi\|^2 = 1$

$$2(\xi - v)\cdot A\xi = 2(\xi - v)\cdot \lambda\xi = 2\lambda(\xi \cdot \xi - v \cdot \xi)$$

$$= 2\lambda(1 - v \cdot \xi) \leq \lambda \varepsilon^2$$

Estimate the other term

$$(\xi - v)\cdot A(v - \xi) \leq \|\xi - v\| \, \|A\| \, \|v - \xi\| = \|A\| \, \|\xi - v\|^2$$

$$\leq \|A\| \varepsilon^2$$

Therefore

$$|\lambda - v^T A v| \lesssim |\lambda (\xi - v) \cdot A\xi| + |(\xi - v) \cdot A(v - \xi)|$$

$$\lesssim \quad \lambda \varepsilon^2 + \|A\| \varepsilon^2 \simeq (\lambda + \|A\|) \varepsilon^2$$

Actually

$$|\lambda - v^T A v| \lesssim (\lambda + \|A\|) \|v - \xi\|^2$$

ε^2 from the Raleigh quotient

... and then one more ε by using μ in the inverse iteration as the current best approximation for λ.

This gives Cubic convergence for Raleigh Quotient iteration.