Let $A \in \mathbb{R}^{n \times n}$

Spectral Theorem: If $A^T = A$ then the eigenvalues of A are real and have an orthonormal basis of eigenvectors.

λ_i - eigenvalues $\qquad \lambda_i \in \mathbb{R}$

ξ_i - eigenvectors

$$\xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$A\xi_i = \lambda_i \xi_i$$

$$S = \left[\xi_1 \middle| \xi_2 \middle| \dots \middle| \xi_n \right]$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

then $A = SDS^{-1}$

When the eigenvectors are orthonormal then S is orthogonal, that is $S^TS = I$. In other words $S^{-1} = S^T$

$$S^TS = \begin{bmatrix} \xi_1^T \\ \hline \xi_2^T \\ \hline \vdots \\ \hline \xi_n^T \end{bmatrix} \left[\xi_1 \middle| \xi_2 \middle| \dots \middle| \xi_n \right] = \begin{bmatrix} \xi_1 \cdot \xi_1 & \xi_1 \cdot \xi_2 & \dots & \xi_1 \cdot \xi_n \\ \xi_2 \cdot \xi_1 & \xi_2 \cdot \xi_2 & & \\ \vdots & & & \\ \xi_n \cdot \xi_1 & \dots & & \xi_n \cdot \xi_n \end{bmatrix} = I$$

When A is symmetric then $A = QDQ^T$ where Q is an orthogonal matrix.

Recall from last time $H = I - 2vv^T$ where

$$v = \frac{a_1 - ce_1}{2v^Ta_1} = \frac{a_1 - ce_1}{\|a_1 - ce_1\|}$$

and $c = \pm\|a_1\|$

chose the sign so the denominator was as large as possible to reduce rounding error.

```
julia> using LinearAlgebra

julia> A=rand(4,4)
4×4 Matrix{Float64}:
 0.0197464  0.294243  0.0642386  0.46501
 0.441981   0.738674  0.640188   0.413124
 0.84054    0.358429  0.495009   0.309332
 0.95816    0.54411   0.0197887  0.848024

julia> a1=A[:,1]
4-element Vector{Float64}:
 0.019746429278822752
 0.4419808827072651
 0.8405404022056417
 0.9581595949313164
```

```
julia> c=norm(a1)
1.349190497760388

julia> e1=[1,0,0,0]
4-element Vector{Int64}:
 1
 0
 0
 0

julia> norm(a1-c*e1)
1.894029199616119

julia> norm(a1+c*e1)
1.9219556155096862
```

```
julia> v1=(a1+c*e1)/norm(a1+c*e1)
4-element Vector{Float64}:
 0.7122625080372527
 0.22996414648735555
 0.43733601099978453
 0.49853367434669954

julia> H=I-2*v1*v1'
4×4 Matrix{Float64}:
 -0.0146358  -0.32759   -0.622996  -0.710174
 -0.32759     0.894233  -0.201143  -0.22929
 -0.622996   -0.201143   0.617474  -0.436053
 -0.710174   -0.22929   -0.436053   0.502928
```

leading to the QR factorization

```
julia> H*A
4×4 Matrix{Float64}:
 -1.34919      -0.856     -0.533101  -0.937098
  4.55576e-17   0.367301   0.447329  -0.0395664
  1.02163e-16  -0.347831   0.128236  -0.551576
  8.5655e-17   -0.26098   -0.398307  -0.133352

julia> H*A*H'
4×4 Matrix{Float64}:
  1.29779   -0.00138617   1.09217    0.915599
 -0.370909   0.247548     0.219587  -0.299177
  0.42577   -0.210366     0.389663  -0.253567
  0.428341  -0.122683    -0.135302   0.166457
```

seems useless ...

Algorithm 26.1. Householder Reduction to Hessenberg Form

for $k = 1$ to $m-2$

$x = A_{k+1:m,k}$

$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$

$v_k = v_k / \|v_k\|_2$

$A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$

$A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m} v_k) v_k^*$

$K=1$

$x = A[2:m, 1]$

$c = \pm \| x \|$

$$v = \frac{x - ce_1}{\|x - ce_1\|}$$

choose $\pm$ to make denominator large.

```
julia> x=A[2:4,1]
3-element Vector{Float64}:
 0.4419808827072651
 0.8405404022056417
 0.9581595949313164

julia> c=norm(x)
1.349045988014367

julia> e1=[1,0,0]
3-element Vector{Int64}:
 1
 0
 0

julia> norm(x-c*e1)
1.5643992720417477

julia> norm(x+c*e1)
2.1982618653713413
```

```
julia> v=(x+c*e1)/norm(x+c*e1)
3-element Vector{Float64}:
 0.8147468229036867
 0.3823659116534114
 0.4358714537266739
```

```
julia> v1=[0; v]
4-element Vector{Float64}:
 0.0
 0.8147468229036867
 0.3823659116534114
 0.4358714537266739
```

```
julia> H=I-2*v1*v1'
4×4 Matrix{Float64}:
  1.0   -0.0        -0.0        -0.0
 -0.0   -0.327625   -0.623063   -0.71025
 -0.0   -0.623063    0.707593   -0.333325
 -0.0   -0.71025    -0.333325    0.620032

julia> H'*A*H
4×4 Matrix{Float64}:
  0.0197464    -0.466699   -0.292876   0.0579229
 -1.34905       1.27148     0.464245   0.205509
  1.63195e-16   0.389636    0.309733   0.0948204
  5.57062e-17   0.387145   -0.281773   0.500492
```

```
julia> H*A*H'
4×4 Matrix{Float64}:
  0.0197464    -0.466699  -0.292876  0.0579229
 -1.34905       1.27148    0.464245  0.205509
  1.63195e-16   0.389636   0.309733  0.0948204
  5.57062e-17   0.387145  -0.281773  0.500492

julia> H*A*H
4×4 Matrix{Float64}:
  0.0197464    -0.466699  -0.292876  0.0579229
 -1.34905       1.27148    0.464245  0.205509
  1.63195e-16   0.389636   0.309733  0.0948204
  5.57062e-17   0.387145  -0.281773  0.500492
```

note $H = I - 2vv^T$ is symmetric, so

HAH^T

HAH

are all the same...

by induction do the same with the smaller matrix...

eventually one has a matrix that's not upper triangular but in Hessenberg form...

```
julia> Q,H=hessenberg(A)
Hessenberg{Float64, UpperHessenberg{Float64, Ma
r{Float64}, Bool}
Q factor:
4×4 LinearAlgebra.HessenbergQ{Float64, Matrix{F
 1.0   0.0        0.0        0.0
 0.0  -0.327625   0.942592  -0.0646735
 0.0  -0.623063  -0.267006  -0.735187
 0.0  -0.71025   -0.20057    0.674772
H factor:
4×4 UpperHessenberg{Float64, Matrix{Float64}}:
  0.0197464  -0.466699   0.166932   0.247518
 -1.34905     1.27148   -0.474171  -0.181434
   ⋅         -0.54927    0.311026  -0.283075
   ⋅           ⋅         0.0935186  0.499199
```

note $Q = H_1 H_2$ is no longer symmetric...

$\boxed{A = QHQ^T}$

or

~~$A = Q^T H Q$~~

Since A is related to H by a similarity transform then the eigenvalues of A are the same as the eigenvalues of H.

```
julia> opnorm(A-Q*H*Q')
3.3337289353506e-16

julia> opnorm(A-Q'*H*Q)
2.5875928713407097
```

```
julia> eigvals(A)
4-element Vector{ComplexF64}:
 -0.35837631340834875 + 0.0im
   0.3104861301842152 - 0.07903437853855023im
   0.3104861301842152 + 0.07903437853855023im
     1.83885724846253 + 0.0im

julia> eigvals(Matrix(H))
4-element Vector{ComplexF64}:
 -0.35837631340834875 + 0.0im
   0.3104861301842152 - 0.07903437853855023im
   0.3104861301842152 + 0.07903437853855023im
     1.83885724846253 + 0.0im
```

Finding the eigenvalues of H is easier because of all the zeros in the lower left corner.