5. Given $A \in \mathbf{R}^{m\times n}$ let $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ be unit vectors such that

$$\|Ax\| = \|A\| \qquad \text{and} \qquad \|A^T y\| = \|A^T\|.$$

$\|A\| = \max\{\|Ax\| : \|x\| = 1\}$ if maximizing over all those x's there is one (or more) where the maximum is attained.

$\|A^T\| = \max\{\|A^T y\| : \|y\| = 1\}$ choose y where you get the maximum.

Thus x and y have been chosen to be vectors for which the maximum in the definition of the matrix 2-norm is attained.

(i) Use the Cauchy–Schwarz inequality to prove that

$$\|Ax\|^2 \le \|A^T Ax\| \qquad \text{and} \qquad \|A^T y\|^2 \le \|AA^T y\|.$$

Cauchy-Schwartz inequality $u \cdot v = \|u\|\,\|v\|\cos\theta \le \|u\|\,\|v\|$

actually $|u \cdot v| \le \|u\|\,\|v\|$

turn that into a dot product

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T Ax = x^T A^T Ax = x \cdot A^T Ax$$
$$\le \underbrace{\|x\|}_{=1}\,\|A^T Ax\| = \|A^T Ax\|$$

$\boxed{\textbf{(ii)}}$ By repeated applications of the definition of the norm it follows that

$$\|AA^T y\| \le \|A\|\|A^T\| \qquad \text{and} \qquad \|A^T Ax\| \le \|A^T\|\|A\|.$$

Explain why $\|AA^T y\| = \|A^T Ax\|$ and finally why $\|A\| = \|A^T\|$.

Instead of doing this ... let's go to question 6.

$x = \xi$ is where the maximum is attained...

6. Let $A \in \mathbf{R}^{m\times n}$ set $\boxed{B = A^T A}$ and define

$$\lambda = \max\{\, x^T B x : \|x\| = 1 \,\}. = \xi^T B \xi$$

(i) Explain why $\lambda \geq 0$ and show that $\|A\| = \|A^T\| = \sqrt{\lambda}$.

(ii) Choose $\boxed{\xi \in \mathbf{R}^n}$ to be a unit vector such that $\boxed{\xi^T B \xi = \lambda}$ and show that $\|B\xi\| \leq \lambda$.

(iii) Expand the inner product $(B\xi - \lambda\xi)^T(B\xi - \lambda\xi)$ and show that $\|B\xi - \lambda\xi\| = 0$.

(iv) Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.

recall the definition of $\|A\|$

$$\|A\| = \max\{\, \|Ax\| : \|x\| = 1 \,\}$$

looks similar to

$$\lambda = \max\{\, x^T B x : \|x\| = 1 \,\}.$$

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T Ax = x^T \underbrace{A^T A}_{B} x = x \cdot A^T A x$$

$$\|Ax\|^2 = x^T B x$$

$$\lambda = \max\{\, x^T B x : \|x\| = 1 \,\} = \max\{\, \|Ax\|^2 : \|x\| = 1 \,\}$$

flesh this out and draw conclusions...

(ii)

$\boxed{\xi^T B \xi = \lambda}$ and show that $\|B\xi\| \leq \lambda$.

$$\|B\xi\|^2 = B\xi \cdot B\xi = \xi^T B^T B \xi \quad ?$$

$$\|B\xi\| = \|A^T A \xi\| \leq \boxed{?}$$

$$\lambda = \xi^T B \xi = \xi \cdot B\xi \leq \|\xi\| \, \|B\xi\| = \|B\xi\|$$

$$\lambda \leq \|B\xi\|$$

Try this again

unit vectors

$$\|B\xi\| = \|A^T A \xi\| \leq \|A^T\| \, \|A\xi\| \leq \underbrace{\|A^T\|}_{\sqrt{\lambda}} \, \underbrace{\|A\|}_{\sqrt{\lambda}} \, \underbrace{\|\xi\|}_{1} = \lambda$$

Therefore $\|B\xi\| \leq \lambda$ (Probably equality)

$$\|B\xi - \lambda\xi\|^2 =$$

same

$$(B\xi - \lambda\xi)^T(B\xi - \lambda\xi) = (B\xi)^T B\xi - (B\xi)^T \lambda\xi - \lambda\xi^T B\xi + \lambda^2 \xi^T \xi$$

$$= B\xi \cdot B\xi - B\xi \cdot \lambda\xi - \lambda\xi \cdot B\xi + \lambda^2 \xi \cdot \xi$$

proved earlier ... definition ... unit vector

$$= \|B\xi\|^2 - 2\lambda\xi \cdot B\xi + \lambda^2 \underbrace{\|\xi\|^2}_{= 1}$$

$$\leq \lambda^2 - 2\lambda\xi \cdot B\xi + \lambda^2$$

$$= 2(\lambda^2 - \lambda\xi \cdot B\xi) = 2(\lambda^2 - \lambda^2) = 0$$