HW1 from First Steps in Numerical Analysis (Due Sept 16)

- Step 1 Exercise 1
- Step 2 Exercise 2abcd
- Step 3 Exercises 1, 3, 4
- Step 4 Exercises 1abef, 2abef
- Step 5 Exercises 1abc, 2abc, 3, 5

Step 1

1. Calculate the period of a simple pendulum of length 75 cm, given that g is 981 cm/s^2.

The period of a simple pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{75}{981}}$$

also note there is model error in this formula for T

When carrying out the calculation, first divide

These numbers are approximations only accurate to the digits shown

$75/981 \approx 0.07645260$

```
julia> 75/981
0.0764525993883792
```

rounding error

then take square root $\sqrt{\frac{75}{981}} \approx 0.2765006$

```
julia> sqrt(0.07645260)
0.276500632910668
```

rounding error

generated error because $\sqrt{\cdot}$ is only an approximation.

initial error

```
julia> pi
π = 3.1415926535897...
```

3.141593

```
julia> 2*3.141593
6.283186
```

6.283186

propagated error

```
julia> 6.283186*0.2765006
1.7373046989115999
```

generated error

rounding error

$T \approx 1.737305$

Step 2

2. For each of the following numbers:

34.78219, 3.478219, 0.3478219, 0.03478219,

(a) chop to three significant digits ($3S$),
(b) chop to three decimal places ($3D$),
(c) round to three significant digits ($3S$),
(d) round to three decimal places ($3D$).

(a)	34.78219,	3.478219,	0.3478219,	0.03478219,
	34.7	3.47	0.347	0.0347
(b)	34.78219,	3.478219,	0.3478219,	0.03478219,
	34.782	3.478	0.347	0.034

(c) 34.78219, 3.478219, 0.3478219, 0.03478219,

34.8 3.48 0.348 0.0348

(d) 34.78219, 3.478219, 0.3478219, 0.03478219,

34.782 3.478 0.348 0.035

Step 3

1. 8.24 + 5.33.

$e_{abs} \le 0.005 + 0.005 = 0.01$

$8.24 + 5.33 = 13.57 \pm 0.01 \in [13.56, 13.58]$

Thus we know sum is 13.6 is correct to all shown digits and not any more.

see here

Should have been #3

2. 124.53 – 124.52.

$e_{abs} \le 0.005 + 0.005 = 0.01$

$124.53 - 124.52 = 0.01 \pm 0.01 \in [0.00, 0.02]$

Thus we know the difference is 0.0 correct to all shown digits.

wrong problem

(3) $e_{rel} \le \frac{0.005}{4.27} + \frac{0.005}{3.13} \approx 0.00277$

$e_{abs} \le 0.00277(4.27 \times 3.13) = 0.037$

$4.27 \times 3.13 = 13.365 \pm 0.037 \in [13.328, 13.402]$

Thus, the product is 13 correct to all shown digits

4. 9.48 × 0.513 – 6.72.

First consider the product

$e_{rel} \le \frac{.005}{9.48} + \frac{0.0005}{0.513} \approx 0.0015$

```
julia> 0.005/9.48+0.0005/0.513
0.0015020850297332643
```

Convert the relative error back to absolute error

$$e_{abs} = (9.48 \times 0.513)\, e_{rel} \approx (9.48 \times 0.513)(0.0015) \approx 0.0073$$

```
julia> (9.48*0.513)*0.0015
0.00729486
```

Now consider the error propagated by the sum.

$$e_{abs} \leq 0.0073 + 0.005 \approx 0.0123$$

$$9.48 \times 0.513 - 6.72 = -1.8568 \pm 0.0123 \in [-1.8691, -1.8445]$$

```
julia> 9.48*0.513-6.72
-1.8567599999999995
```

as the first 8 in -1.8568 could round either up or down, we only know the answer is -2 one significant digit.

Step 4

1. Evaluate the following using three-digit decimal normalized floating point arithmetic with rounding:
(a) $6.19 \times 10^2 + 5.82 \times 10^2$.
(b) $6.19 \times 10^2 + 3.61 \times 10^1$.
(e) $(3.60 \times 10^3) \times (1.01 \times 10^{-1})$.
(f) $(-7.50 \times 10^{-1}) \times (-4.44 \times 10^1)$.

(a) $6.19 \times 10^2 + 5.82 \times 10^2 = 1201 \xrightarrow[\text{rounding}]{} 1.20 \times 10^3$

```
julia> 6.19e2+5.82e2
1201.0
```

(b) $6.19\times10^{2}+3.61\times10^{1} = 655.1 \xrightarrow[\text{rounding}]{} 6.55\times10^{2}$

```
julia> 6.19e2+3.61e1
655.1
```

(e) $(3.60\times10^{3})\times(1.01\times10^{-1}) = 363.6 \xrightarrow[\text{rounding}]{} 3.64\times10^{2}$

```
julia> 3.60e3*1.01e-1
363.6
```

(f) $(-7.50\times10^{-1})\times(-4.44\times10^{1}) = 33.3 \xrightarrow[\text{rounding}]{} 3.33\times10^{1}$

```
julia> -7.50e-1*-4.44e1
33.3
```

2. Estimate the accumulated errors in the results of Exercise 1, assuming that all values are correct to $3S$.

(a) $6.19\times10^{2}+5.82\times10^{2}$.
(b) $6.19\times10^{2}+3.61\times10^{1}$.
(e) $(3.60\times10^{3})\times(1.01\times10^{-1})$.
(f) $(-7.50\times10^{-1})\times(-4.44\times10^{1})$.

(a) Propagated ~~Initial~~ error

$$e_{abs} \lesssim 0.005\times10^{2} + 0.005\times10^{2} \doteq 1\times10^{0}$$

Generated error

$$e_{abs} = |1201 - 1.20\times10^{3}| = 1\times10^{0}$$

accumulated error

$$e_{abs} \lesssim 1\times10^{0} + 1\times10^{0} = 2\times10^{0}$$

(b) ~~Initial~~ Propagated error

$$e_{abs} \le 0.005\times10^{2} + 0.005\times10^{1} = 5.5\times10^{-1}$$

Generated error

$$e_{abs} = |655.1 - 6.55\times10^{2}| = 1\times10^{-1}$$

Accumulated error

$$e_{abs} \le 5.5\times10^{-1} + 1.0\times10^{-1} = 6.5\times10^{-1}$$

(e) ~~Initial~~ Propagated error

$$e_{rel} \le \frac{0.005}{3.60} + \frac{0.005}{1.01} = 0.00634$$

```
julia> 0.005/3.60+0.005/1.01
0.006339383939383939384
```

Generated error

$$e_{abs} = |363.6 - 3.64\times10^{2}| = 0.6$$

$$e_{rel} = \frac{0.6}{363.6} = 0.00165$$

```
julia> 0.6/363.6
0.0016501650165016502
```

Accumulated error

$$e_{rel} \le 0.00634 + 0.00165 = 0.00799$$

$$e_{abs} \le (363.6)\,e_{rel} = 2.91$$

```
julia> 363.6*0.00799
2.9051640000000005
```

(f) ~~Initial~~ Propagated error

$$e_{rel} \leq \frac{0.005}{7.50} + \frac{0.005}{4.44} \approx 0.00179$$

```
julia> 0.005/7.50+0.005/4.44
0.0017927927927927929
```

Generated error

$$e_{abs} \approx |33.3 - 3.33\times10^{1}| = 0$$

$$e_{rel} \approx 0$$

Accumulated error

$$e_{rel} \lesssim 0.00179 + 0 = 0.00179$$

Step 5

1. Find the Taylor series expansions about $x = 0$ for each of the following functions.

(a) $\cos x$.

(b) $1/(1-x)$.

(c) e^x.

For each series also determine a general remainder term.

(a)

$$f(x) = \cos x \qquad f(0) = \cos 0 = 1$$

$$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$$

$$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1$$

$$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0$$

$$\vdots \qquad \vdots$$

$$f^{(2l)}(0) = (-1)^l$$

$$f^{(2l+1)}(0) = 0$$

Taylor series is

$$f(h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(0) + \int_0^h \frac{1}{n!}(h-t)^n f^{(n+1)}(t)\,dt$$

since $f^{(k)}(0)=0$ when k is odd, we any sum that ends with n even could as well end at $n+1$ odd. Thus, we assume n is odd and write $n = 2m+1$. It follows

$$\sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(0) = \sum_{l=0}^{m} \frac{h^{2l}}{(2l)!} f^{(2l)}(0) = \sum_{l=0}^{m} \frac{(-1)^l h^{2l}}{(2l)!}$$

Similarly

$$\int_0^h \frac{1}{n!}(h-t)^n f^{(n+1)}(t)\,dt = \int_0^h \frac{(h-t)^{2m+1}}{(2m+1)!}(-1)^{m+1}\cos t$$

thus

$$\cos h = \sum_{l=0}^{m} \frac{(-1)^l h^{2l}}{(2l)!} + \int_0^h \frac{(h-t)^{2m+1}}{(2m+1)!}(-1)^{m+1}\cos t$$

the Lagrange form of the remainder is

$$\int_0^h \frac{(h-t)^{2m+1}}{(2m+1)!}(-1)^{m+1}\cos t = \frac{h^{2m+2}}{(2m+2)!}(-1)^{m+1}\cos\xi$$

for some ξ between 0 and h.

(b) $f(x) = (1-x)^{-1}$ $\quad f(0) = 1$

$f'(x) = (1-x)^{-2}$ $\quad f'(0) = 1$ $\quad f^{(k)}(x) = k!(1-x)^{-k-1}$

$f''(x) = 2(1-x)^{-3}$ $\quad f''(0) = 2$ $\quad f^{(k)}(0) = k!$

$f'''(x) = 3!(1-x)^{-4}$ $\quad f'''(0) = 3!$

Therefore

$$(1-h)^{-1} = \sum_{k=0}^{n} \frac{k!}{k!} h^k + \int_0^h \frac{1}{n!} (h-t)^n (n+1)! (1-t)^{-n-2} dt$$

$$= \sum_{k=0}^{n} h^k + \int_0^h (n+1) \frac{(h-t)^n}{(1-t)^{n+2}} dt$$

If $h<1$ then

$$\int_0^h (n+1) \frac{(h-t)^n}{(1-t)^{n+2}} dt = \frac{h^{n+1}}{1-h}$$

Thus, rather than the Lagrange form of the remainder we simply write

$$\frac{1}{1-h} = \sum_{k=0}^{n} h^k + \frac{h^{n+1}}{1-h}$$

Note that this formula is also known as the geometric series and can be derived from long division.

(c) $f(x) = e^x$ $\quad f(0) = 1$

$f^{(k)}(x) = e^x$ $\quad f^{(k)}(0) = 0$ $\quad$ for all $k \geq 0$

Therefore

$$e^h = \sum_{k=0}^{n} \frac{1}{k!} h^k + \int_0^h \frac{1}{n!}(h-t)^n e^t \, dt$$

The Lagrange form of the remainder is

$$e^h = \sum_{k=0}^{n} \frac{1}{k!} h^k + \frac{1}{(n+1)!} h^{(n+1)} e^{\xi}$$

for some ξ between 0 and h

(a) $\cos x$.
(b) $1/(1-x)$.
(c) e^x.

For each series also determine a general remainder term.

2. For each of the functions in Exercise 1, evaluate $f(0.5)$ using a calculator and by using the first four terms of your Taylor expansion.

(a) $$\cos h \approx \sum_{l=0}^{m} \frac{(-1)^l h^{2l}}{(2l)!}$$ where $h = 0.5$ and $m = 3$.

Therefore

$$\cos 0.5 \approx \sum_{l=0}^{3} \frac{(-1)^l (0.5)^{2l}}{(2l)!} \approx 0.8775825$$

```
julia> a(l)=(-1)^l*(0.5)^(2*l)/factorial(2*l)
a (generic function with 1 method)

julia> sum(a(l) for l=0:3)
0.8775824652777777
```

By calculator

$\cos 0.5 \approx 0.8775826$

```
julia> cos(0.5)
0.8775825618903728
```

(b) $\dfrac{1}{1-h} \approx \displaystyle\sum_{k=0}^{n} h^k$ where $h=0.5$ and $n=3$.

Therefore

$$\frac{1}{1-0.5} \approx \sum_{k=0}^{3} (0.5)^k = 1.875$$

By calculator

$$\frac{1}{1-0.5} = 2.0$$

```
julia> sum((0.5)^k for k=0:3)
1.875

julia> 1/(1-0.5)
2.0
```

(c) $e^h \approx \displaystyle\sum_{k=0}^{n} \frac{1}{k!} h^k$ where $h=0.5$ and $n=3$.

Therefore

$$e^{0.5} \approx \sum_{k=0}^{3} \frac{1}{k!}(0.5)^k = 1.645833$$

By calculator

$$e^{0.5} \approx 1.648721$$

```
julia> a(k)=1/factorial(k)*(0.5)^k
a (generic function with 1 method)

julia> sum(a(k) for k=0:3)
1.6458333333333333

julia> exp(0.5)
1.6487212707001282
```

3. Use the remainder term found in Exercise 1(c) to find the value of n required in the Taylor series for $f(x) = e^x$ about $x = 0$ to give 5D accuracy for all x between 0 and 1.

The remainder term is

$$R_n = \frac{1}{(n+1)!} h^{(n+1)} e^{\xi} \quad \text{where } \xi \text{ is between } 0 \text{ and } h.$$

Since we want the approximation to hold for all $h \in [0,1]$ then ξ must also satisfy $\xi \in [0,1]$. It follows that

$$|R_n| \le \max_{h\in[0,1]} \max_{\xi\in[0,h]} \frac{1}{(n+1)!} h^{(n+1)} e^{\xi} = \frac{1}{(n+1)!} 1^{(n+1)} e^{1} = \frac{e}{(n+1)!}.$$

For five digit accuracy we need $|R_n| \le 0.000005$. This is guaranteed by the Taylor theorem provided

$$\frac{e}{(n+1)!} \le 0.000005 = 5\times 10^{-6}$$

solving for n by guess and check yields that $n = 9$.

```
julia> R(n)=exp(1)/factorial(n+1)
R (generic function with 1 method)

julia> [(n,R(n)-5e-6) for n=1:10]
10-element Vector{Tuple{Int32, Float64}}:
 (1, 1.3591359142295225)
 (2, 0.45304197140984087)
 (3, 0.11325674285246021)
 (4, 0.02264734857049204)
 (5, 0.0037703914284153406)
 (6, 0.0005343416326307629)
 (7, 6.241770407884537e-5)
 (8, 2.4908560087605956e-6)
 (9, -4.250914399123941e-6)
 (10, -4.931901309011268e-6)
```

first time $R_n \lesssim 5\times10^{-6}$

5. Evaluate $P(3.1)$ and $P'(3.1)$, where $P(x) = x^3 - 2x^2 + 2x + 3$, using the technique of nested multiplication.

Generally we want

$$p(x) = x(x(x-2)+2)+3$$

and since $p'(x) = 3x^2 - 4x + 2$ then

$$p'(x) = x(3x-4)+2$$

Using the recurrence in the book we have

$$\begin{aligned}
p_0 &= a_n & q_0 &= 0 \\
p_1 &= p_0\bar{x} + a_{n-1} & q_1 &= q_0\bar{x} + p_0 \\
&= a_n\bar{x} + a_{n-1} & &= a_n \\
p_2 &= p_1\bar{x} + a_{n-2} & q_2 &= q_1\bar{x} + p_1 \\
&= a_n\bar{x}^2 + a_{n-1}\bar{x} + a_{n-2} & &= 2a_n\bar{x} + a_{n-1} \\
&\vdots & &\vdots \\
p_n &= P(\bar{x}) & q_n &= P'(\bar{x})
\end{aligned}$$

Where $a_0 = 3$, $a_1 = 2$, $a_2 = -2$ and $a_3 = 1$. Thus

```
julia> n=3
       using OffsetArrays
       A=OffsetVector([3,2,-2,1],0:n);
       P=OffsetVector([A[n],zeros(3)...],0:n);
       Q=OffsetVector(zeros(4),0:n);
```

This stores the coefficients of the polynomial in A and initializes the vectors P and Q for the recurrence. Note that the OffsetArrays package is used because vectors usually start with index 1.

The the recurrence can be computed as

```
julia> x=3.1
       for k=1:n
           P[k]=P[k-1]*x+A[n-k]
           Q[k]=Q[k-1]*x+P[k-1]
       end
       println("p($x)=",P[n])
       println("p'($x)=",Q[n])
p(3.1)=19.771
p'(3.1)=18.43
```

This shows that

$$p(3.1) = 19.771$$

and

$$p'(3.1) = 18.43.$$