7. a. Using $A = [0.007\ \ -0.8; -0.1\ \ 10]$ and $b = (0.7, 10)^T$ from Eq. (2.3), find the solution of $Ax = b$ in ~~MATLAB~~ (use `y=A\b`).

b. Find the residual vector r, the norm of the residual, and the absolute error (recall that the true solution is $x_1 = -1500$, $x_2 = -14$).

c. Compute the ratio of the absolute error to the norm of the residual $\|y - x\| / \|r\|$.

7a.

```
julia> A=[0.007 -0.8; -0.1 10]
2×2 Matrix{Float64}:
  0.007   -0.8
 -0.1    10.0

julia> b=[0.7,10]
2-element Vector{Float64}:
  0.7
 10.0

julia> y=A\b
2-element Vector{Float64}:
 -1500.0000000000002
   -14.000000000000002
```

7b.

```
julia> using LinearAlgebra

julia> r=A*y-b
2-element Vector{Float64}:
 1.1102230246251565e-15
 0.0

julia> norm(r)
1.1102230246251565e-15

julia> x=[-1500,-14]
2-element Vector{Int32}:
 -1500
   -14

julia> norm(y-x)
2.273806142312601e-13
```

7c.

```
julia> norm(y-x)/norm(r)
204.80624990463548
```

8. a. Solve $Ax = b$ with $A = [0.0002\ \ 2; 2\ \ 2]$ and $b = (2, 4)^T$ using three-digit arithmetic and naive Gaussian elimination. Compare this to the solution found using exact arithmetic.

b. Repeat part (a) using Gaussian elimination with partial pivoting.

8a. The augmented matrix is

$$\left[\begin{array}{cc|c} 0.0002 & 2 & 2 \\ 2 & 2 & 4 \end{array}\right]$$

Gaussian elimination with 3-digit rounding yields

$$\left[\begin{array}{cc|c} 0.0002 & 2 & 2 \\ 2 & 2 & 4 \end{array}\right] \quad r_2 \leftarrow r_2 - \frac{2}{0.0002} r_1$$

$2 - 20000 = -19998 \approx -20000$
$4 - 20000 = -19996 \approx -20000$

$$\left[\begin{array}{cc|c} 0.0002 & 2 & 2 \\ 0 & -20000 & -20000 \end{array}\right] \quad r_1 \leftarrow r_1 + \frac{2}{20000} r_2$$

$$\left[\begin{array}{cc|c} 0.0002 & 0 & 0 \\ 0 & -20000 & -20000 \end{array}\right] \quad \text{Solution is } x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Exact solution

$$\left[\begin{array}{cc|c} 0.0002 & 2 & 2 \\ 0 & -19998 & -19996 \end{array}\right] \quad r_1 \leftarrow r_1 + \frac{2}{19998} r_1$$

$$2 - 2\left(\frac{19996}{19998}\right) = 2\left(\frac{2}{19998}\right)$$

$$\left[\begin{array}{cc|c} 0.0002 & 0 & \frac{4}{19998} \\ 0 & -19998 & -19996 \end{array}\right] \quad \text{Solution is } x = \begin{bmatrix} \frac{20000}{19998} \\ \frac{19996}{19998} \end{bmatrix} = \begin{bmatrix} \frac{10000}{9999} \\ \frac{9998}{9999} \end{bmatrix}$$

In Julia the exact solution can be calculated using fractions as

```
julia> A=[2//10000 2; 2 2]
2×2 Matrix{Rational{Int32}}:
 1//5000  2//1
 2//1     2//1

julia> b=[2,4]
2-element Vector{Int32}:
 2
 4

julia> A\b
2-element Vector{Rational{Int32}}:
 10000//9999
  9998//9999
```

8b. Using partial pivoting one has

$$\left[\begin{array}{cc|c} 0.0002 & 2 & 2 \\ 2 & 2 & 4 \end{array}\right] \quad r_1 \leftrightarrow r_2$$

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 0.0002 & 2 & 2 \end{array}\right] \quad r_2 \leftarrow r_2 - 0.0001\, r_1 \qquad \begin{array}{l} 2 - 0.0002 = 1.9998 \approx 2 \\ 4 - 0.0004 = 1.9996 \approx 2 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & 2 & 2 \end{array}\right] \quad r_1 \leftarrow r_1 - r_2$$

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 2 & 2 \end{array}\right] \quad \text{Solution} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note that the solution with partial pivoting is much closer to the exact solution.

5. a. The matrix $A = [0\ \ 0\ \ 1; 0\ \ 2\ \ 3; 1\ \ 4\ \ -4]$ can't be row-reduced without row interchanges, so our LU decomposition method does not apply. Show that if $P = [0\ \ 0\ \ 1; 0\ \ 1\ \ 0; 1\ \ 0\ \ 0]$, then PA has an LU decomposition. Identify L and U explicitly, and then write them in the form of Example 2.2.4.

b. Repeat with $A = [0\ \ 1\ \ 1; 0\ \ 2\ \ 3; 1\ \ 4\ \ -4]$ and the same P.

5a:

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

amazingly PA is already upper triangular. Therefore

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the compact form of the LU factorization discussed in example 2.2.4 is

$$\begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

5b.

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Now reduce PA using elimination steps

$$\begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad r_3 \leftarrow r_3 - \frac{1}{2} r_2 \qquad 1 - \frac{3}{2} = -\frac{1}{2}$$

place $\frac{1}{2}$ in the 32 position

$$U = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Therefore the compact form of the LU factorization discussed in example 2.2.4 is

$$\begin{bmatrix} 1 & 4 & -4 \\ 0 & 2 & 3 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

15. Use ~~MATLAB~~ to find the LU decomposition of `ones( 10, 10 )`. Explain why L, U, and P have the form they do in terms of the Gaussian elimination with partial pivoting algorithm.

```
julia> A=ones(10,10)
10×10 Matrix{Float64}:
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```

To allow Julia to factor a singular matrix.

```
julia> L,U=lu(A,check=false)
Failed factorization of type LU{Float64, Matrix{Float64}}

julia> L
10×10 Matrix{Float64}:
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 1.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
```

```
julia> U
10×10 Matrix{Float64}:
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

julia> L*U
10×10 Matrix{Float64}:
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
 1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```

6. a. Use the Cholesky Decomposition Algorithm to find the Cholesky decomposition of the cross-product matrix $A^T A$, where $A = [1\ \ 0\ \ 1; 1\ \ 0\ \ -1; 1\ \ 1\ \ 0; 1\ \ 1\ \ 1]$.

b. Use your Cholesky decomposition from part (a) to solve $A^T A x = A^T b$, where $b = (1, -1, 1, -1)^T$.

6a.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Cholesky Decomposition Algorithm:

1. Set $R_1 = \sqrt{a_{11}}$.
2. Begin loop ($p = 2$ to n):
3. Solve $R_{p-1}^T \gamma_{p-1} = c_{p-1}$ for γ_{p-1} by forward substitution.
4. Set $r_{pp} = \sqrt{a_{pp} - \gamma_{p-1}^T \gamma_{p-1}}$.
5. Set $R_p = \begin{pmatrix} R_{p-1} & \gamma_{p-1} \\ 0 & r_{pp} \end{pmatrix}$.
6. End loop.
7. Set $L = R_n^T$.

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Step 1: $R_1 = \sqrt{4} = 2$

Step 2: $p = 2$

Step 3: $2\gamma_1 = 2$, $\gamma_1 = 1$

$$\begin{array}{r} 16 \\ 3 \\ \hline 48 \end{array}$$

Step 4: $r_{22} = \sqrt{2 - 1} = 1$

Step 5: $R_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

Step 2: $p = 3$

Step 3: $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \gamma_2 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

Step 4: $r_{33} = \sqrt{3 - \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}} = \sqrt{3 - \frac{1}{2}} = \sqrt{5/2}$

Step 5: $R = R_3 = \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & \sqrt{5/2} \end{bmatrix}$

Check that $A^TA = R_3^TR_3$

```
julia> R=[2 1 1/2; 0 1 1/2; 0 0 sqrt(5/2)]
3×3 Matrix{Float64}:
 2.0  1.0  0.5
 0.0  1.0  0.5
 0.0  0.0  1.58114

julia> R'*R
3×3 Matrix{Float64}:
 4.0  2.0  1.0
 2.0  2.0  1.0
 1.0  1.0  3.0
```

$= A$

6b.

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A^TAx = R^T\underbrace{Rx}_{y} = A^Tb$$

$$\begin{cases} R^Ty = A^Tb \\ Rx = y \end{cases}$$

$$R^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & \sqrt{5/2} \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & \sqrt{5/2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & \sqrt{5/2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$y_1 = 0$
$y_2 = 0$
$y_3 = \sqrt{2/5}$

$$\begin{bmatrix} 2 & 1 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & \sqrt{5/2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2/5} \end{bmatrix}$$

$x_1 = 0$
$x_2 = -1/5$
$x_3 = 2/5$

Solution $x = \begin{bmatrix} 0 \\ -1/5 \\ 2/5 \end{bmatrix}$

10. a. Prove that if every leading principal minor of a symmetric matrix A is positive definite, then A is positive definite.

b. Prove that the diagonal elements of a positive definite matrix are positive.

c. Prove that every leading principal minor of a positive definite matrix has a positive determinant.

10a. The leading principal minors A_p are the $p \times p$ submatrices in the northwest corner of the matrix. If A is symmetric than it's square, in which case A is an $n \times n$ matrix for some n. In this case $A = A_n$ for $p = n$. Now since A_n is positive definite than so is A.

10b. Since A is positive definite it has a cholesky factorization given as $A = R^T R$ moreover, since A is invertibe, then no column of R could be identically zero. It follows that

$$a_{kk} = (R^T R)_{kk} = \sum_{i=1}^{n} (R^T)_{ki} R_{ik} = \sum_{i=1}^{n} (R_{ik})^2 > 0$$

since this is a sum of non-negative terms at least one of which is not zero.

10c. Since a matrix A is positive definite if and only if all leading principle minor submatrices are also positive definite, it is enough to show the determinant of a positive definite matrix is positive. Let R be the Cholesky factorization such that $A = R^T R$. Then

$$\det A = \det R^T R = (\det R^T)(\det R) = (\det R)^2 \geq 0$$

Since A is invertible then $\det A \neq 0$.

It follows that $\det A > 0$.

2. a. The matrix norm induced by the l_1 vector norm is $\|A\|_\gamma = \max_{j=1}^{n}\{\sum_{i=1}^{n} |a_{ij}|\}$, the maximum column sum. The matrix norm induced by the l_∞ vector norm is $\|A\|_\rho = \max_{i=1}^{n}\{\sum_{j=1}^{n} |a_{ij}|\}$, the maximum row sum. For $A = [1\ \ 2\ \ 3; 4\ \ 5\ \ 6; 7\ \ 8\ \ 9]$, find (by hand) $\|A\|_F$, $\|A\|_\gamma$, $\|A\|_\rho$.

b. Find $\|A\|_s$ by finding the largest eigenvalue of $A^T A$ (use ~~`max(eig(A'*A))`~~).

maximum(eigvals(A'*A))

2a.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

By definition

$$\|A\|_\gamma = \max\left\{ \sum_{i=1}^{n} |a_{ij}| : j = 1,2,\dots,n \right\}$$

$$= \max\{1+4+7,\ 2+5+8,\ 3+6+9\}$$

$$= \max\{12, 15, 18\} = 18$$

$$\|A\|_\rho = \max\left\{ \sum_{j=1}^{n} |a_{ij}| : i = 1,2,\dots,n \right\}$$

$$= \max\{1+2+3,\ 4+5+6,\ 7+8+9\}$$

$$= \max\{6, 15, 24\} = 24$$

$$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2}$$

$$= \sqrt{1+4+9+16+25+36+49+64+81} = \sqrt{285} \approx 16.88194$$

```
julia> A=[1 2 3; 4 5 6;7 8 9]
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 7  8  9

julia> A.*A
3×3 Matrix{Int64}:
  1   4   9
 16  25  36
 49  64  81

julia> sum(A.*A)
285

julia> sqrt(sum(A.*A))
16.881943016134134
```

2b. $\|A\|_s = \sqrt{\max\{\lambda : \lambda \text{ is an eigenvalue of } A^tA\}}$

Using Julia we have

```
julia> A=[1 2 3;4 5 6;7 8 9]
3×3 Matrix{Int32}:
 1  2  3
 4  5  6
 7  8  9

julia> B=A'*A
3×3 Matrix{Int32}:
 66   78   90
 78   93  108
 90  108  126

julia> using LinearAlgebra

julia> lambda=eigvals(B)
3-element Vector{Float64}:
  -1.516797508290897e-14
   1.1414134196298527
 283.8585865803701
```

Thus

$\|A\|_s \approx \sqrt{283.8586} \approx 16.84810$

```
julia> maximum(lambda)
283.8585865803701

julia> sqrt(maximum(lambda))
16.848103352614206
```

10. **a.** An orthogonal matrix satisfies $U^T U = I$. Show that $\|Ux\|_2 = \|x\|_2$ for all x if U is orthogonal. (*Hint:* Consider the quadratic form $x^T Ax$ with $A = U^T U$.)

b. Show that an orthogonal matrix U has $\|U\|_2 = 1$ and $\kappa_s(U) = 1$.

10a.
$$\|Ux\|_2 = \sqrt{Ux \cdot Ux} = \sqrt{(Ux)^T Ux} = \sqrt{x^T U^T U x} = \sqrt{x^T I x}$$
$$= \sqrt{x^T x} = \sqrt{x \cdot x} = \|x\|_2$$

10b.
$$\|U\|_2 = \max\{\|Ux\|_2 : \|x\|_2 = 1\} = \max\{\|x\|_2 : \|x\|_2 = 1\} = 1.$$

Since orthogonal matrices are square and $U^T U = I$, then $U^T = U^{-1}$. Consequently $UU^T = UU^{-1} = I$ implies that U^T is also an orthogonal matrix. Thus

$$\|U^{-1}\|_2 = \|U^T\|_2 = 1.$$

It follows that

$$\kappa_s(U) = \|U\|_2 \|U^{-1}\|_2 = 1.$$