Math 466/666: Homework Assignment 5

This homework uses the vector and matrix 2-norms to explore some properties of matrices, eigenvalues and eigenvectors. Note that you will need to use Julia or similar software for one of these problems.

Given a matrix $A \in \mathbf{R}^{m\times n}$ recall that the spectral matrix norm is defined as

$$\|A\|_s = \max\left\{ \|Ax\|_2 : \|x\|_2 = 1 \right\}$$

where the vector 2-norms $\|Ax\|_2$ and $\|x\|_2$ are given by

$$\|Ax\|_2 = \sqrt{(Ax)^T Ax} \qquad \text{and} \qquad \|x\|_2 = \sqrt{x^T x} \qquad \text{for} \qquad x \in \mathbf{R}^n.$$

For convenience of notation we will, henceforth, drop the subscripts and write

$$\|A\| = \|A\|_s, \qquad \|Ax\| = \|Ax\|_2 \qquad \text{and} \qquad \|x\| = \|x\|_2.$$

1. Show that $\|I\| \geq 1$ where I is the identity operator.

2. Let $A \in \mathbf{R}^{n\times n}$ be invertible. Show that $\|A\|\|A^{-1}\| \geq 1$.

3. Define

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}.$$

Use the definition of the matrix 2-norm to compute $\|A\|$ and $\|A^T\|$.

4. Let $A \in \mathbf{R}^{7\times 4}$ be a matrix with entries chosen randomly (and independently) from the uniform distribution on the interval $[-1, 1]$. Use any numerical means you prefer to compute $\|A\|$ and $\|A^T\|$ for three different samples of the random matrices just described. For example, in Julia you could enter the commands

```
using LinearAlgebra
A=2*rand(7,4).-1;
opnorm(A)
opnorm(A')
```

three times. What relationship between $\|A\|$ and $\|A^T\|$ do you notice? Please include all program listings and computer output as part of your answer for this question.

5. Given $A \in \mathbf{R}^{m\times n}$ let $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ be unit vectors such that

$$\|Ax\| = \|A\| \qquad \text{and} \qquad \|A^T y\| = \|A^T\|.$$

Thus x and y have been chosen to be vectors for which the maximum in the definition of the matrix 2-norm is attained.

(i) Use the Cauchy–Schwarz inequality to prove that

$$\|Ax\|^2 \le \|A^TAx\| \qquad \text{and} \qquad \|A^Ty\|^2 \le \|AA^Ty\|.$$

(ii) By repeated applications of the definition of the norm it follows that

$$\|AA^Ty\| \le \|A\|\|A^T\| \qquad \text{and} \qquad \|A^TAx\| \le \|A^T\|\|A\|.$$

Explain why $\|AA^Ty\| = \|A^TAx\|$ and finally why $\|A\| = \|A^T\|$.

6. Let $A \in \mathbf{R}^{m\times n}$ set $B = A^TA$ and define

$$\lambda = \max\left\{ x^TBx : \|x\| = 1 \right\}.$$

(i) Explain why $\lambda \ge 0$ and show that $\|A\| = \|A^T\| = \sqrt{\lambda}$.

(ii) Choose $\xi \in \mathbf{R}^n$ to be a unit vector such that $\xi^TB\xi = \lambda$ and show that $\|B\xi\| \le \lambda$.

(iii) Expand the inner product $(B\xi - \lambda\xi)^T(B\xi - \lambda\xi)$ and show that $\|B\xi - \lambda\xi\| = 0$.

(iv) Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.

7. [Extra Credit and for Math 666] Let $B \in \mathbf{R}^{n\times n}$ be any symmetric matrix with $B^T = B$ and choose $\xi \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ such that $\lambda = \max\{ x^TBx : \|x\| = 1 \}$ and $\xi^TB\xi = \lambda$.

(i) Find an example of a matrix B such that $\lambda < 0$.

(ii) Show for any $v \in \mathbf{R}^n$ that $v^TBv \le \lambda v^Tv$.

(iii) Set $v = \xi + \epsilon w$ where $w \in \mathbf{R}^n$ and $\epsilon > 0$ and simplify to prove that

$$2w^T(B\xi - \lambda\xi) \le \epsilon(\lambda w^Tw - w^TBw).$$

(iv) Now set $w = B\xi - \lambda\xi$. Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.