Interval arithmetic:

Idea is to use intervals to represent exact bounds on numeric approximations

$3.45 + 4.87 - 5.16$

$x^* = 3.45$ then $x \in (3.445, 3.455)$

assume properly rounded approximation

$y^* = 4.87$ then $y \in (4.865, 4.875)$

$z^* = 5.16$ then $z \in [5.155, 5.165]$

even

$-z^* = -5.16$ $\qquad -z \in [-5.165, -5.155]$

In what interval is the correct answer $x+y-z$.

$$x+y-z \in (3.445 + 4.865 - 5.165, 3.455 + 4.875 - 5.155) = (3.145, 3.175)$$

```
julia> 3.445+4.865-5.165
3.1450000000000005

julia> 3.455+4.875-5.155
3.175
```

To how many significant digits do I know the correct answer from the approximations x^*, y^* and z^*.

Since $x+y-z \in (3.145, 3.175)$ then how many significant digits are represented by this interval?

Maybe 3S? try it round 3.145 → 3.14 ← different so I can't say for what the 3rd digit is.

round 3.175 → 3.18

Maybe 2S? try it round 3.145 → 3.1 ← different so I can't say for what the 2nd digit is.

round 3.175 → 3.2

Maybe 1S? try it round 3.145 → 3

round 3.175 → 3

Only know $\underset{\pm .005}{3.45} + \underset{\pm .005}{4.87} - \underset{\pm .005}{5.16} = 3$ to 1S

Do same problem using the rules for propagated error

We have this bound:

$$e_{abs}(x+y) = |x+y - (x^*+y^*)|$$

$$\le |x-x^*| + |y-y^*| = e_{abs}(x) + e_{abs}(y)$$

Thus,

$3(\pm .005) = \pm .015$

$\boxed{\text{Propagated error...}}$

$e_{abs}((5.16+6.29) - 1.31) \le 0.015.$

```
julia> 3.45+4.87-5.16
3.16

julia> 3.16-0.015
3.145

julia> 3.16+0.015
3.1750000000000003
```

Thus the answer is 3.16 ± 0.015

$x+y-z \in (3.16 - 0.015,\ 3.16 + 0.015)$

$x+y-z \in (3.145,\ 3.75)$ are the same...

from before

Since $x+y-z \in (3.145, 3.175)$ then how many significant digits are represented by this interval?

How errors are propagated through multiplication:

$x \in \mathbb{R}$ and x^* is an approximation of x

$y \in \mathbb{R}$ and y^* is an approximation of y.

$e_{abs}(x) = |x - x^*|$ $\qquad$ $e_{abs}(y) = |y - y^*|$

Propagated error

What is $e_{abs}(xy)$?

exact mult of the approximate values for x and y.

$$e_{abs}(xy) = |xy - x^* y^*|$$

$$= |xy - xy^* + xy^* - x^*y^*|$$

$$= |x(y - y^*) + (x - x^*)y^*|$$

$$\leq |x||y - y^*| + |x - x^*||y^*|$$

make symmetric

$$\leq |x|\, e_{abs}(y) + |y^*|\, e_{abs}(x)$$

$$\leq |x|\, e_{abs}(y) + |y^* - y + y|\, e_{abs}(x)$$

$$\leq |x|\, e_{abs}(y) + (\underbrace{|y^* - y|}_{e_{abs}(y)} + |y|)\, e_{abs}(x)$$

$$\leq |x|\, e_{abs}(y) + |y|\, e_{abs}(x) + e_{abs}(x)\, e_{abs}(y)$$

measure propagated error relatively

$$e_{rel}(xy) = \frac{e_{abs}(xy)}{|xy|} =$$

def. of relative error

$$\leq \frac{|x|\, e_{abs}(y) + |y|\, e_{abs}(x) + e_{abs}(x)\, e_{abs}(y)}{|xy|}$$

$$\lesssim \frac{e_{abs}(y)}{|y|} + \frac{e_{abs}(x)}{|x|} + \frac{e_{abs}(x)\, e_{abs}(y)}{|xy|}$$

Thus

$$e_{rel}(xy) \lesssim \underbrace{\frac{e_{abs}(y)}{|y|}}_{e_{rel}(y)} + \underbrace{\frac{e_{abs}(x)}{|x|}}_{e_{rel}(x)} + \underbrace{\frac{e_{abs}(x)\, e_{abs}(y)}{|xy|}}_{\text{higher order term if } e_{abs} \text{ is small compared to } x \text{ or } y}$$

approximately

$$\boxed{e_{rel}(xy) \lesssim e_{rel}(x) + e_{rel}(y)}$$

relative errors propagate additively through multiplication.

Example: 3.55×2.73

If $x^* = 3.55$? then $x \in (3.545,\ 3.555)$

$y^* = 2.73$ then $y \in (2.725,\ 2.735)$

$$e_{rel}(x) = \frac{e_{abs}(x)}{|x|} \approx \frac{e_{abs}(x)}{|x^*|} \leq \frac{0.005}{|3.55|}$$

$$e_{rel}(y) = \frac{e_{abs}(y)}{|y|} \approx \frac{e_{abs}(y)}{|y^*|} \leq \frac{0.005}{|2.73|}$$

Propagated error

$$e_{rel}(xy) \lesssim e_{rel}(x) + e_{rel}(y) \lesssim \frac{0.005}{|3.55|} + \frac{0.005}{|2.73|}$$

$$\simeq 0.00323995...$$

```
julia> 0.005*(1/3.55+1/2.73)
0.0032399525357271835
```

$$e_{abs}(xy) \approx |xy|\, e_{rel}(xy) \approx |x^* y^*|\, e_{rel}(xy) \approx 0.0314$$

```
julia> 3.55*2.73*(0.0032399525357271835)
0.0314
```

$$xy \in \left(x^*y^* - 0.0314,\ x^*y^* + 0.0314 \right) = \left(9.6601,\ 9.7229 \right)$$

```
julia> 3.55*2.73-0.0314
9.6601

julia> 3.55*2.73+0.0314
9.7229
```

To how many significant digits do I know the correct answer?

Maybe 3S? $9.6601 \xrightarrow{\text{round}} 9.66$ Not that

$9.7229 \rightsquigarrow 9.72$

Maybe 2S? $9.6601 \xrightarrow{\text{round}} 9.7$ ok... same

$9.7229 \rightsquigarrow 9.7$

I know that $xy \approx 9.7$ to 2S.