eneral

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$

$$R_n(x) = \int_{x_0}^{x} \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt$$

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$.
provided f has $n+1$ derivatives.

Assume $x \geq x_0$ (the case $x \leq x_0$ is similar but has more minus signs in it ...at home)

$$R_n(x) = \int_{x_0}^{x} \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt$$

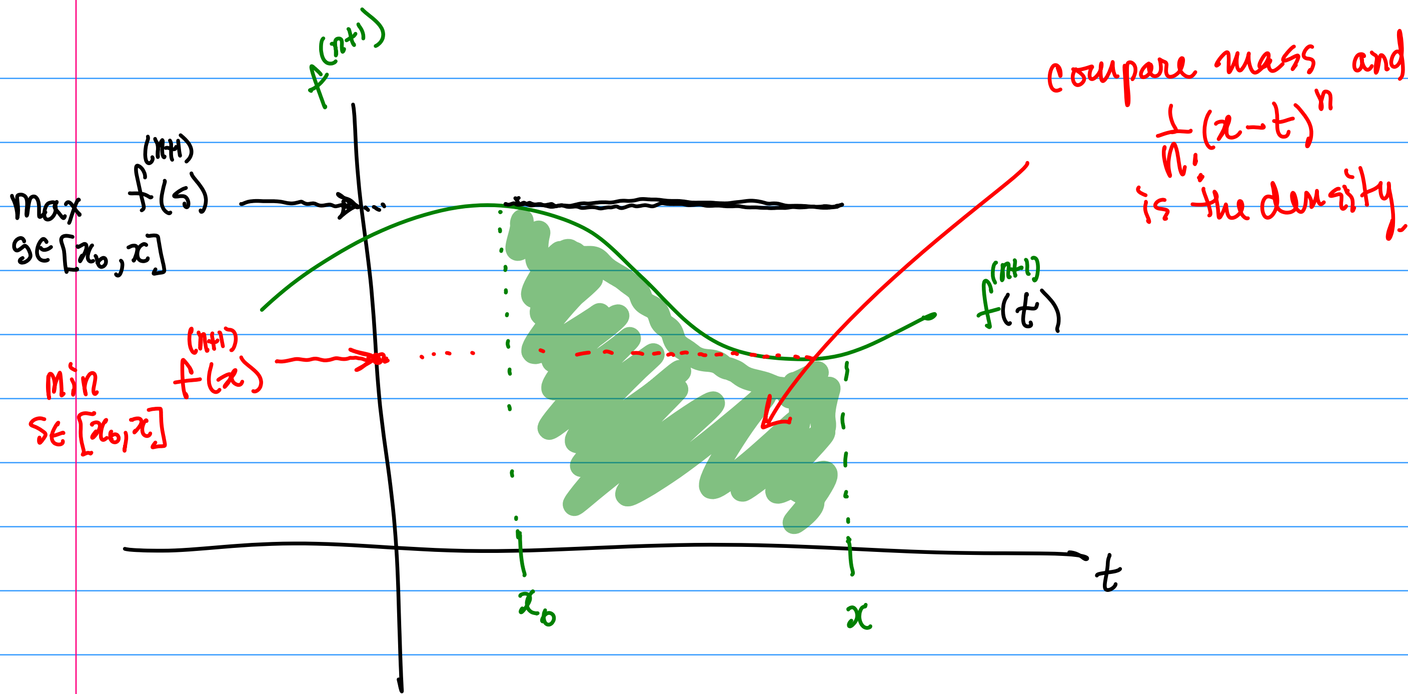
note then $t \in [x_0, x]$ and so $x - t \geq 0$

this term is positive...

$$R_n(x) \leq \int_{x_0}^{x} \frac{1}{n!}(x-t)^n \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) dt$$

$$R_n(x) \geq \int_{x_0}^{x} \frac{1}{n!}(x-t)^n \min_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) dt$$

what about this?



$$\int_{x_0}^{x} \frac{1}{n!}(x-t)^n \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right)$$

$$= \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \int_{x_0}^{x} \frac{1}{n!}(x-t)^n \, dt$$

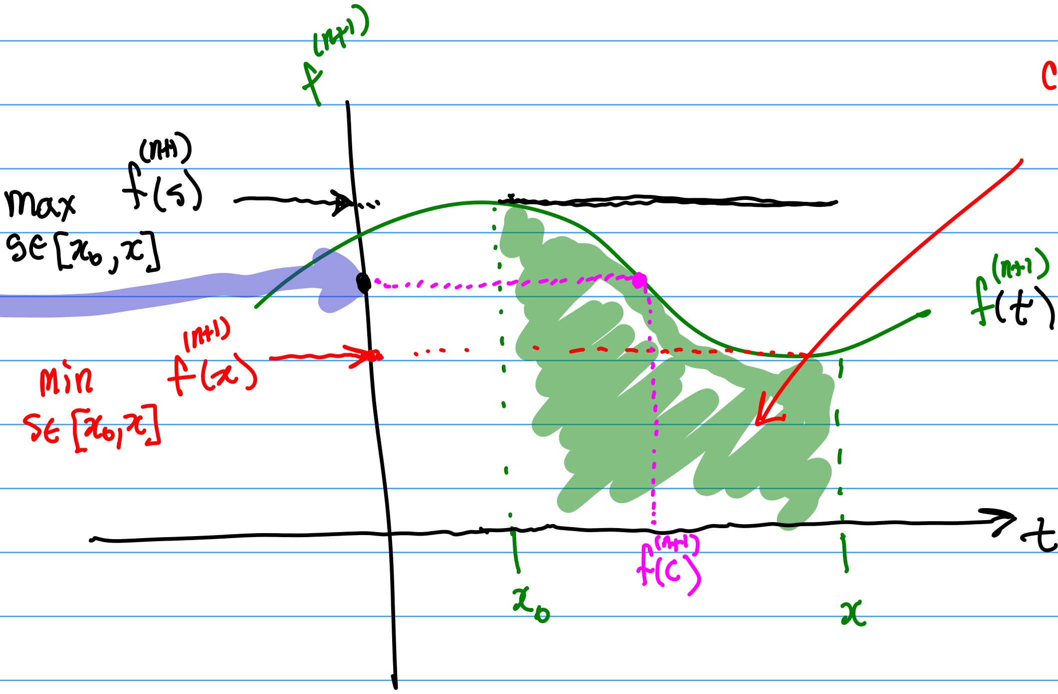
$$= \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \frac{-1}{(n+1)!}(x-t)^{n+1}\Big|_{x_0}^{x}$$

$$= \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \frac{1}{(n+1)!}(x-x_0)^{n+1}$$

$$\min_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \frac{1}{(n+1)!}(x-x_0)^{n+1} \le R_n(x) \le \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \frac{1}{(n+1)!}(x-x_0)^{n+1}$$

$$\min_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right) \le \frac{R_n(x)}{\frac{1}{(n+1)!}\,(x-x_0)^{n+1}} \le \max_{s\in[x_0,x]}\left(f^{(n+1)}(s)\right)$$

some number between the min and max of f



Therefore

$$\frac{R_n(x)}{\frac{1}{(n+1)!}\,(x-x_0)^{n+1}} = f^{(n+1)}(c) \quad \text{for some } c \text{ between } x_0 \text{ and } x.$$

Or

$$R_n(x) = f^{(n+1)}(c)\,\frac{1}{(n+1)!}\,(x-x_0)^{n+1}$$

for some c between x_0 and x.

6. Find the Taylor series expansion about $x = 0$ for each of the following functions:

(i) $\sin x$

(ii) $\sqrt{1-x}$

(iii) e^{2x}

For each series determine a general remainder term.

7. Use the remainder term in Question 6(iii) to find the degree n of the Taylor polynomial approximation to e^{2x} that gives 4D accuracy for all x between 0 and 1.

Example for (i) instead...

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + R_{2n+1}(x)$$

$$R_{2n+1}(x) = \int_0^x \frac{1}{(2n+1)!}(x-t)^{2n+1} f^{(2n+2)}(t)\,dt$$

$$R_n(x) = \int_{x_0}^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt$$

$(\sin x)' = \cos x$

$(\sin x)'' = -\sin x$

$(\sin x)''' = -\cos x$

$$\frac{d^k}{dx^k}\sin x = \begin{cases} \sin x & \text{if } k = 0 \mod 4 \\ \cos x & \text{if } k = 1 \quad " \\ -\sin x & \text{if } k = 2 \quad " \\ -\cos x & \text{if } k = 3 \quad " \end{cases}$$

$k = 4m$

$k = 4m+1$

$k = 4m+2$

$k = 4m+3$

Note

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + R_{2n+2}(x)$$

$$R_{2n+2}(x) = \int_0^x \frac{1}{(2n+2)!}(x-t)^{2n+2} f^{(2n+3)}(t)\,dt$$

$$R_{2n+1}(x) = \int_0^x \frac{1}{(2n+1)!}(x-t)^{2n+1} f^{(2n+2)}(t)\,dt$$

$$= \int_0^x \frac{1}{(2n+1)!}(x-t)^{2n+1} (\pm \sin(t))\,dt$$

$u = \sin(t)$ $\qquad du = \cos t$

$dv = \frac{1}{(2n+1)!}(x-t)^{2n+1}\,dt$ $\qquad v = \frac{-1}{(2n+2)!}(x-t)^{2n+2}$

$$= \pm\left(\sin(t)\frac{-1}{(2n+2)!}(x-t)^{2n+2}\Big|_0^x + \int_0^x \frac{1}{(2n+2)!}(x-t)^{2n+2}\cos t\,dt \right)$$

$$= \pm\int_0^x \frac{1}{(2n+2)!}(x-t)^{2n+2}\cos t\,dt = R_{2n+2}(x)$$

This verifies that we have...

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)}{(2n+1)!}x^{2n+1} + R_{2n+2}(x)$$

$$R_{2n+2}(x) = \int_0^x \frac{1}{(2n+2)!}(x-t)^{2n+2} f^{(2n+3)}(t)\,dt$$

that gives 4D accuracy for all x between 0 and 1.

$$|R_{2n+2}(x)| \leq .00005$$

Solve for the smallest n such that the above holds

Find n so that

$$\left| \int_0^x \frac{1}{(2n+2)!} (x-t)^{2n+2} \cos t \, dt \right| \leq .00005$$

If $\left| \int_0^x \frac{1}{(2n+2)!} (x-t)^{2n+2} \cdot 1 \, dt \right| \leq .00005$ then

what I want also hold true.

$$\left| \frac{1}{(2n+3)!} x^{2n+3} \right| \leq .00005$$

need this forall $x \in [0,1]$

The largest x can be is 1. So the above is guaranteed to hold if

$$\frac{1}{(2n+3)!} \leq .00005 = 5 \times 10^{-5}$$

Check

n	$\frac{1}{(2n+3)!}$
1	.0083...
2	.000198...
3	$2.7557... \times 10^{-6} \leq 5 \times 10^{-5} = 50 \times 10^{-6}$

$$T(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$n=3$ work, but that wasn't the degree of the poly.

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

The degree of the needed polynomial is 7.