```
a=-pi/2; b=0.0
for n=1:5
    c=b-(b-a)/(f(b)-f(a))*f(b)
    fc=f(c)
    println("[$a,$b] $c $fc")
    b=a; b=c
end
```

x_2

Write as a sequence x_n of approximations...

$x_0 = a$, $x_1 = b$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$\boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)}$$

2 Effectiveness and the secant method

Let α be the solution so $f(\alpha) = 0$. Define the error in x_n as $e_n = \alpha - x_n$. Estimate:

$$e_{n+1} = \alpha - x_{n+1} \approx \alpha - \left(x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \right)$$

$$= \alpha - x_n + \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\approx e_n + \frac{x_n - \alpha + \alpha - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\approx e_n + \frac{-e_n + e_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$= \frac{e_n\left(f(x_n) - f(x_{n-1})\right) + (-e_n + e_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{\cancel{e_n f(x_n)} - e_n f(x_{n-1}) - \cancel{e_n f(x_n)} + e_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\approx \frac{e_{n-1} f(x_n) - e_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Now use Taylor series and other estimates...

$e_n = \alpha - x_n.$ $\qquad x_n = \alpha - e_n$

also $\qquad x_{n-1} = \alpha - e_{n-1}$

$$e_{n+1} = \frac{e_{n-1} f(\alpha - e_n) - e_n f(\alpha - e_{n-1})}{f(\alpha - e_n) - f(\alpha - e_{n-1})}$$

Taylor expansions

$$f(\alpha - e_n) \approx f(\alpha) - e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) - \cdots$$

$$f(\alpha - e_{n-1}) \approx f(\alpha) - e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2} f''(\alpha) - \cdots$$

Denominator: Note $f(\alpha) = 0$

$$f(\alpha - e_n) - f(\alpha - e_{n-1}) \approx \cancel{f(\alpha)} - e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) - \cdots$$
$$- \left(\cancel{f(\alpha)} - e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2} f''(\alpha) - \cdots \right)$$

$$\simeq (-e_n + e_{n-1}) f'(\alpha) + \frac{1}{2}(e_n^2 - e_{n-1}^2) f''(\alpha) - \cdots$$

difference of squares.
$(e_n - e_{n-1})(e_n + e_{n-1})$

$$= (-e_n + e_{n-1}) \left[f'(\alpha) - \frac{1}{2}(e_n + e_{n-1}) f''(\alpha) + \cdots \right]$$

Numerator:

$$e_{n-1} f(\alpha - e_n) - e_n f(\alpha - e_{n-1}) \approx e_{n-1} \left(\cancel{f(\alpha)} - e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) - \frac{e_n^3}{3!} f'''(\alpha) + \cdots \right)$$
$$- e_n \left(\cancel{f(\alpha)} - e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2} f''(\alpha) - \frac{e_{n-1}^3}{3!} f'''(\alpha) + \cdots \right)$$

$$= -e_{n-1}e_n \cancel{f'(\alpha)} + \frac{e_{n-1}e_n^2}{2} f''(\alpha) - e_{n-1}\frac{e_n^3}{3!} f'''(\alpha)$$

$$+ e_n e_{n-1} \cancel{f'(\alpha)} - \frac{e_n e_{n-1}^2}{2} f''(\alpha) + e_n \frac{e_{n-1}^3}{3!} f'''(\alpha)$$

$$\approx \frac{1}{2}\left(e_{n-1}e_n^2 - e_n e_{n-1}^2\right) f''(\alpha) + \frac{1}{6}\left(-e_{n-1}e_n^3 + e_n e_{n-1}^3\right) f'''(\alpha) + \cdots$$

$$\approx e_{n-1}e_n \left[\frac{1}{2}(e_n - e_{n-1}) f''(\alpha) + \frac{1}{6}\underbrace{\left(-e_n^2 + e_{n-1}^2\right)}_{-(e_n - e_{n-1})(e_n + e_{n-1})} f'''(\alpha) + \cdots\right]$$

$$\approx e_{n-1}e_n \frac{1}{2}(e_n - e_{n-1})\left[f''(\alpha) - \frac{1}{3}\left(e_n + e_{n-1}\right) f'''(\alpha) + \cdots\right]$$

Therefore

$$e_{n+1} = \frac{e_{n-1}e_n \frac{1}{2}\cancel{(e_n - e_{n-1})}\left[f''(\alpha) - \frac{1}{3}\left(e_n + e_{n-1}\right) f'''(\alpha) + \cdots\right]}{-\cancel{(e_n - e_{n-1})}\left[f'(\alpha) - \frac{1}{2}(e_n + e_{n-1}) f''(\alpha) + \cdots\right]}$$

$$e_{n+1} \approx \frac{\boxed{-\frac{1}{2}e_{n-1}e_n \left[f''(\alpha)\right.} - \frac{1}{3}\left(e_n + e_{n-1}\right) f'''(\alpha) + \cdots\left.\right]}{\boxed{f'(\alpha)} - \frac{1}{2}(e_n + e_{n-1}) f''(\alpha) + \cdots}$$

$$e_{n+1} \approx -\left[\frac{f''(\alpha)}{2f'(\alpha)}\right] e_{n-1}e_n$$

Suppose $x_n \to \alpha$ then $e_n \to 0$ & $e_{n-1} \to 0$

Question how fast does it converge?

$e_{n+1} \sim e_{n-1} e_n$. $\quad e_{n+1} \approx K e_{n-1} e_n$

Suppose $e_n \approx M e_{n-1}^k$ $\quad e_{n+1} \approx M e_n^k \approx M\left(M e_{n-1}^k\right)^k$

$\approx M^{1+k} e_{n-1}^{k^2}$

Plug it in

$$M^{1+k} e_{n-1}^{k^2} \approx K e_{n-1} M e_{n-1}^k$$

Solve for k by observing power of k on both sides has to be the same., So it holds for all the values of e_{n-1}

Thus $\quad k^2 = k+1$

$$k^2 - k - 1 = 0$$

$a=1 \quad b=-1 \quad c=-1$

choose + solution since $k>0$

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$k \approx (1+\sqrt{5})/2 \approx 1.618.$$

This is a consistency result. If

$$e_n \approx M e_{n-1}^{k}$$

then $k \approx 1.618$

Suppose $e_{n-1} \approx 0.5 \times 10^{-m}$ for m decimal digits of accuracy.

$$e_n \approx M \left(0.5 \times 10^{-m}\right)^{1.618}$$

$$\approx \underbrace{M}_{\text{about } 1}\, 0.5 \times 10^{-1.618m}$$

60% more decimal digits of accuracy...