Newtons method:

Trying to solve $f(x)=0$. Let's say the exact solution is α. So $f(\alpha)=0$

Suppose x_n is an approximation of α.

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \underbrace{\tfrac{1}{2}(\alpha - x_n)^2 f''(c_n)}_{\text{error term}}$$

for some c_n between α and x_n

Consider the approximation

$$f(\alpha) \approx f(x_n) + (\alpha - x_n) f'(x_n)$$

Since $f(\alpha)=0$ then

$$f(x_n) + (\alpha - x_n) f'(x_n) \approx 0$$

Now solve for α

$$(\alpha - x_n) f'(x_n) \approx -f(x_n)$$

$$\alpha - x_n \approx \frac{-f(x_n)}{f'(x_n)}$$

$$\boxed{\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)}}$$

Use this as a rule for an iterative scheme

$$x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

improved approximation .... iterate.

and hope $x_n \to \alpha$.

Error analysis...

$e_n = \alpha - x_n$

$e_{n+1} = \alpha - x_{n+1}$ ...

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(c_n)$$

Thus,

$$0 = \frac{f(x_n)}{f'(x_n)} + \frac{e_n f'(x_n)}{f'(x_n)} + \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}$$

$$e_{n+1} = \alpha - x_{n+1} = \alpha - \left[x_n - \frac{f(x_n)}{f'(x_n)} \right]$$

$$= \underbrace{\alpha - x_n}_{e_n} + \frac{f(x_n)}{f'(x_n)} = \cancel{e_n} + \left[\cancel{-e_n} - \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)} \right]$$

$$e_{n+1} = -\frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)} \approx \frac{-e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$e_{n+1} \approx \frac{-e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

for secant method the exponent here was $k = \frac{1+\sqrt{5}}{2} \approx 1.618$

interpreted this as 61.8 % more sig. digits each iteration

The 2 exponent means double the number of significant digits at each iteration...

```
julia> xn=x0
2

julia> xn=xn-f(xn)/df(xn)
2.5221887802138703

julia> xn=xn-f(xn)/df(xn)
2.5425691666820014

julia> xn=xn-f(xn)/df(xn)
2.542641356856975 7

julia> xn=xn-f(xn)/df(xn)
2.542641357773526 5

julia> xn=xn-f(xn)/df(xn)
2.5426413577735265
```

Number of digits which match nearly doubles at each iteration.