Backwards error analysis:

Example: Quadratic equation

$$ax^2 + bx + c = 0$$

Let $f(x) = ax^2 + bx$ note $\boxed{f(0) = 0.}$

for any equation
$g(x) = 0$
could set
$f(x) = g(x) - g(0)$

Suppose x^* is an approximation of the true solution α.

$$f(\alpha) = c \quad \text{and} \quad f(x^*) \approx c$$

Define $c^* = f(x^*)$ and the residual error as $|c - c^*|$

Question: How to determine the error in x^* from the residual error..

Note if $c = c^*$ then probably x^* was correct...
exact...

Taylor's theorem:

$$f(\alpha) = f(x^*) + \overbrace{(\alpha - x^*)}^{\text{error I'm interested in}} f'(\xi)$$

for ξ between α and x^*

Since $f(\alpha) = c$ and $f(x^*) = c^*$

$$c = c^* + (\alpha - x^*) f'(\xi)$$

$$c - c^* = (\alpha - x^*) f'(\xi)$$

$$\alpha - x^* = \frac{c - c^*}{f'(\xi)}$$

$$e_{abs}(x^*) = |\alpha - x^*| = \frac{|c - c^*|}{|f'(\xi)|} = \frac{\text{residual error}}{|f'(\xi)|}$$

We don't know ξ but we do know f'

$f(x) = ax^2 + bx$

$f'(x) = ax + b$

Put some numbers in place

$(x-2)(x+3) = x^2 + x - 6$

solutions $x = 2$ and -3

$\alpha = 2$

$f(x) = x^2 + x$ $\qquad$ $f(2) = 4 + 2 = 6$

$x^* = 2.1$ ← this approximation of α came from some algorithm...

$f'(x) = 2x + 1$

Let's pretend I don't know α is 2.

What's the residual error

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julia> (2.1)^2+2.1
6.51
```

$$f(2.1) = (2.1)^2 + 2.1 = 6.51$$

residual error $|6 - 6.51| = 0.51$

$$|\alpha - 2.1| = \frac{0.51}{|f'(\xi)|}$$

- If $f'(\xi) = 0$ then the residual doesn't say anything...

- If $|f'(\xi)| \approx 1$ then the actual error is about the same as the residual error...

"ill conditioned"

- If $f'(\xi) \approx 0$ but not exactly 0. then it's difficult to say much about $e_{abs}(x^*)$ from the residual error.

I know that ξ is between α and x^* so

let $K = \max \left\{ \frac{1}{|f'(\xi)|} : |\alpha - \xi| \leq |\alpha - x^*| \right\}$

This K gives a definite bound

$$e_{abs}(x^*) \le K \, |c - c^*|$$

residual error...

In the example

$$f'(2.1) = 2x+1 = 2 \cdot 2.1 + 1 = 4.2 + 1 = 5.2$$

$$f'(2) = 2 \cdot 2 + 1 = 5$$

So $|f'(\xi)| \ge 5$ and $K \le \frac{1}{5}$

$$e_{abs}(x^*) \le \frac{1}{5} \; 0.51 = .102$$

bound on $e_{abs}(x^*)$

$$e_{abs}(x^*) = |\alpha - x^*| = |2 - 2.1| = .1$$

What about $e_{rel}(x^*) = \dfrac{|\alpha - x^*|}{|\alpha|}$?

Taylor theorem

$$f(\alpha) = f(0) + (\alpha - 0) f'(\eta)$$

for some η between 0 and α.

Since $f(0) = 0$ and $f(\alpha) = c$. Then

$$c = \alpha f'(\eta)$$

$$\frac{1}{|\alpha|} = \left| \frac{f'(\eta)}{c} \right| \leq \frac{M}{|c|}$$

Let $M = \max \{ |f'(\eta)| : \text{for all } \alpha \text{ between } 0 \text{ and } \alpha \}$

Therefore

$$|\alpha - x^*| \leq K |c - c^*| \quad \text{and} \quad \frac{1}{|\alpha|} \leq M \frac{1}{|c|}$$

α.

It follows

$$e_{rel}(x^*) = \frac{|\alpha - x^*|}{|\alpha|} \leq KM \frac{|c - c^*|}{|c|}$$

relative residual error...

Goal use this idea of backwards error analysis to understand solutions to $Ax = b$.

To do this I need a notion of absolute value that works for vectors and matrices...

$$x \in \mathbb{R}^n \quad \text{then} \quad \|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x \cdot x}$$

What is the norm of a matrix?

recall this:

$$M = \max \{ |f'(\eta)| : \text{for all } \alpha \text{ between } 0 \text{ and } x \}$$

by analogy...

$$\|A\| = \max \{ \|Ax\| : \text{for all } x \in \mathbb{R}^n \text{ with } \|x\| \leq 1 \}$$

matrix norm

vector norm

vector norm

defined matrix norm in terms of things that I already know...

What are the properties of matrix norms?

First since Matrix is defined by vector norms, then we get the same properties. In particular

$$\|A + B\| \leq \|A\| + \|B\|$$

and

$$\|\alpha A\| = |\alpha| \, \|A\|$$

Second more is true...

$$\|Az\| = \frac{\|z\|}{\|z\|}\|Az\| = \left\| \frac{Az}{\|z\|} \right\| \|z\|$$

$$= \left\| A \frac{z}{\|z\|} \right\| \|z\|$$

since $\frac{z}{\|z\|} = 1$

Then

$$\left\| A \frac{z}{\|z\|} \right\| \leq \max \left\{ \|A\underset{q}{x}\| : \text{for all } x \in \mathbb{R}^n \text{ with } \underset{4}{\|x\|} \leq 1 \right\}$$

So

$$\left\| A \frac{z}{\|z\|} \right\| \leq \|A\|$$

and $\|Az\| \leq \|A\| \, \|z\|$

We'll use these matrix norms and backward error analysis to understand approximations to $Ax = b$.