① Recall the difference tables so far required the x's to be equally spaced like

$$x_i = x_0 + hi .$$

② The Lagrange basis

$$L_j(x) = \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)}$$

works for any distinct collection of x's.

Newton's divided difference formula combines the advantages of both methods...

Given $(x_0, f(x_0))$, $(x_1, f(x_1))$, ..., $(x_n, f(x_n))$ we want to find the unique polynomial p of degree n such that $p(x_k) = f(x_k)$ for $k = 0, ..., n$.

Notation:

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\vdots$$

$$f(x_0, \dots, x_n) = \frac{f(x_1, \dots, x_n) - f(x_0, \dots, x_{n-1})}{x_n - x_0}$$

recursively defined compute in a table of differences..

Define basis functions

$$\begin{aligned}
\varphi_0(x) &= 1 \\
\varphi_1(x) &= x - x_0 \\
\varphi_2(x) &= (x-x_0)(x-x_1) \\
\varphi_3(x) &= (x-x_0)(x-x_1)(x-x_2) \\
&\vdots \\
\varphi_n(x) &= (x-x_0)\cdots(x-x_{n-1})
\end{aligned}$$

basis for polynomials of degree n.

Note the Vandermonde matrix corresponding to this basis is triangular.

Then the unique polynomial of degree n passing through the points $(x_k, f(x_k))$ for $k=0,\dots,n$ is

$$p(x) = f(x_0)\varphi_0(x) + f(x_0,x_1)\varphi_1(x) + f(x_0,x_1,x_2)\varphi_2(x) + \cdots + f(x_0,\dots,x_n)\varphi_n(x)$$

Polynomial interpolation theorem: This is the counterpart to the Taylor Theorem for Taylor polynomials....

Idea approximate $f(x) \approx p(x)$ and find the error...

$$E(x) = f(x) - p(x).$$

Since $p(x_k) = f(x_k)$ for $k=0,\dots,n$, then $E(x_k)=0$ for $k=0,\dots,n$.

Estimate $E(t^*)$ for some t^* where $t^* \neq x_k$ for all $k=0,\dots,n$.

Trick: Write $F(t) = E(t) - \alpha\, q(t)$

additional parameter...

where $q(t) = (x-x_0)(x-x_1)\cdots(x-x_n)$

$$F(x_k) = E(x_k) - \alpha q(x_k) = 0 - \alpha 0 = 0$$

Choose α so there is one more 0 at $t = t^*$.

$$F(t^*) = E(t^*) - \alpha q(t^*) = 0 \quad \text{when} \quad \alpha = \frac{E(t^*)}{q(t^*)}$$

Thus define

$$F(t) = E(t) - \frac{E(t^*)}{q(t^*)}\, q(t)$$

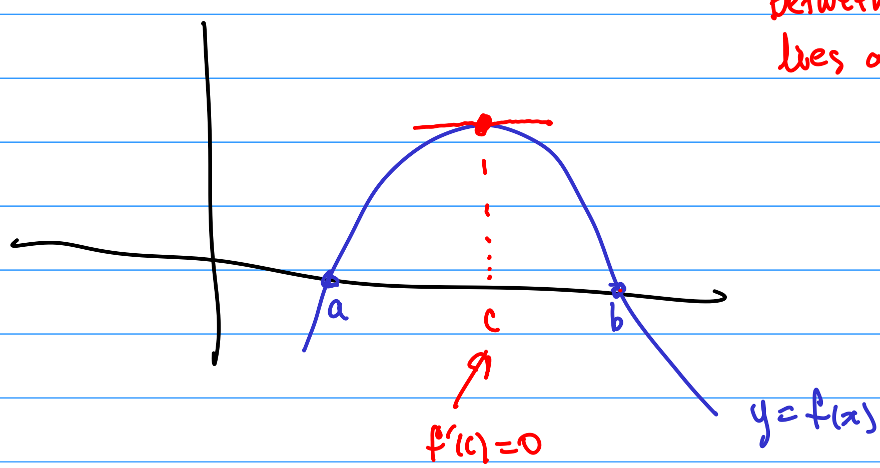
It follows that $F(t^*) = 0$ and $F(x_k) = 0$ for $k = 0, \dots, n$.

one more zero $n+1$ zeros here.

Thus $F(t)$ has $n+2$ zeros

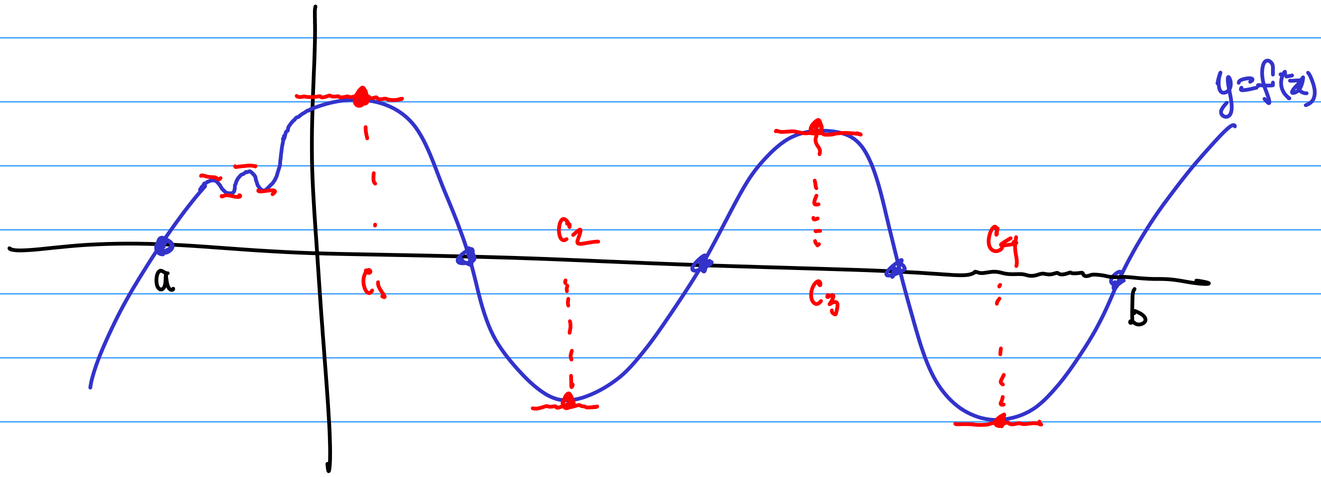
Rolle's Theorem:

between two zeros of f lies a zero of f'



If $f(a)=0$, $f(b)=0$ there is some point c between such that $f'(c)=0$.

Rolle's theorem generalizes to functions w/ more zeros.



If f has 5 zeros between a and b then f' has at least 4 zeros between a and b.

Since $F(t^*)=0$ and $F(x_k)=0$ for $k=0,\dots,n$, then

$$F(t) = E(t) - \frac{E(t^*)}{q(t^*)}\, q(t)$$

$F(t)$ has at least $n+2$ zeros between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$F'(t)$ has at least $n+1$ zeros between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$F''(t)$ has at least n zeros between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$F^{(3)}(t)$ has at least $n-1$ zeros between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$F^{(k)}(t)$ has at least $n-k+2$ zeros between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$\vdots$

$F^{(n+1)}(t)$ has at least 1 zero between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

Let ξ be the zero between $\min(t^*, x_0,\dots,x_n)$ and $\max(t^*, x_0,\dots,x_n)$.

$$F^{(n+1)}(\xi) = E^{(n+1)}(\xi) - \frac{E(t^*)}{q(t^*)} q^{(n+1)}(\xi) = 0$$

$$q^{(n+1)}(\xi) = \frac{d^{n+1}}{dx^{n+1}} \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{\text{polynomial of degree } n+1}\Big|_{x=\xi} = (n+1)!$$

$$E^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \cancel{p^{(n+1)}(\xi)} = f^{(n+1)}(\xi)$$

interpolating polynomial of degree n

Therefore

$$f^{(n+1)}(\xi) - \frac{E(t^*)}{q(t^*)}(n+1)! = 0$$

Solve for $E(t^*)$

$$E(t^*) = \frac{q(t^*)}{(n+1)!} f^{(n+1)}(\xi)$$

error in the approximation . . .

<u>Conclusion:</u>

$$f(t) = p(t) + E(t) = p(t) + \frac{q(t)}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between $\min(t, x_0, \ldots, x_n)$ and $\max(t, x_0, \ldots, x_n)$.

Or

$$f(t) = p(t) + \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

Recall Taylor theorem.

$$f(t) = T_n(t) + R_n \qquad R_n = \frac{(t-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between t and x_0.