```
julia> H1=I-2*v*v'
3×3 Matrix{Float64}:
 -0.0936586  -0.749269  -0.65561
 -0.749269    0.486674  -0.44916
 -0.65561    -0.44916    0.606985
```

Remark... actually forming the matrix H_1 is not a computationally efficient thing to do...

If $A \in \mathbb{R}^{m \times n}$ then $v \in \mathbb{R}^m$

$$\underset{m\times m}{H_1} = \underset{m\times m}{I} - 2\underset{m\times 1}{v}\underset{1\times m}{v^T}$$

In the computing lab $m >> n$.

Thus H_1 has m^2 entries which is rather huge...

Since its a matrix we'll use it to multiply things...

$$H_1 B = H_1 \left[b_1 \middle| b_2 \middle| \cdots \middle| b_p \right] = \left[H_1 b_1 \middle| H_1 b_2 \middle| \cdots \middle| H b_p \right]$$

So really all we need is $H_1 b$ for some vector $b \in \mathbb{R}^m$.

$$H_1 b = (I - 2vv^T) b = b - 2vv^T b = b - 2v(v \cdot b)$$

m subtractions

m multiplications.

Total work is about m operations...
to find $H_1 b$...

Note that forming $H_1 \in \mathbb{R}^{m \times m}$ already takes m^2 operations so don't do that.

The builtin Julia library for QR also doesn't make the H_i matrices, but instead just remembers the vectors v used to make them...

$$Q = \overbrace{H_n H_{n-1} \cdots H_2 H_1}$$

never make the H's and never mult them together to get Q...

Then $\underset{m \times n}{A} = \underset{m \times m}{Q} \; \underset{m \times n}{R}$

Applying H_i to a vector is m operations

After n of those reflectors: Total operations $m \times n$.

So whenever Julia needs to multiply by Q it instead uses $H_n H_{n-1} \cdots H_2 H_1$.

Back to the trapezoid method: Understand how wrong our answer is using Taylor's theorem...

Total error

$$E = \int_a^b f(x)\,dx - \sum_{j=0}^{N-1} h\,\frac{f(x_j)+f(x_{j+1})}{2}$$

$$= \sum_{j=0}^{N-1}\left[\int_{x_j}^{x_{j+1}} f(x)\,dx - h\,\frac{f(x_j)+f(x_{j+1})}{2}\right] = \sum_{j=0}^{N-1} E_j$$

The truncation error

$$E_j = \int_{x_j}^{x_{j+1}} f(x)\,dx - h\,\frac{f(x_j)+f(x_{j+1})}{2}$$

Use Taylors theorem here... $x_j = x_0 + hj$

$$\int_{x_j}^{x_{j+1}} f(x)\,dx = \int_0^1 \boxed{f(x_j+\vartheta h)}\,h\,d\vartheta$$

$$x = x_j + \vartheta h$$
$$dx = h\,d\vartheta$$

Taylor series expand these two terms...

On the other side

$$h\,\frac{f(x_j)+f(x_{j+1})}{2} \to h\,\frac{f(x_j)}{2} + h\,\frac{\boxed{f(x_j+h)}}{2}$$

Book makes these expansions and plugs them in...

$$f_{j+1} = f(x_j + h) = f_j + hf'(x_j) + \boxed{\frac{h^2}{2!} f''(x_j)} + \cdots$$

e trapezoidal form

Taylor's Error term here

$$f(x) = f_j + (x - x_j) f'(x_j) + \underbrace{\boxed{\frac{(x - x_j)^2}{2!} f''(x_j)}}_{R_1} + \cdots$$

exact form

Eventually gets

$$|E_j| \le \frac{1}{12} h^3 \max_{x_j \le x \le x_{j+1}} |f''(x)|$$

Thus...

$$|E| \le \sum_{j=0}^{N-1} |E_j|$$

$$\le \frac{N}{12} h^3 \max_{a \le x \le b} |f''(x)| = \frac{(b-a)h^2}{12} \max_{a \le x \le b} |f''(x)|$$

$$f(x) = f(x_j) + (x - x_j) f'(x_j) + R_1.$$

In general

approx of $f(x)$ $\quad T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} (x - x_0)^k f^{(k)}(x_0)$

error $\quad R_n(x) = \int_{x_0}^{x} \frac{1}{n!} (x - t)^n f^{(n+1)}(t)\, dt$

$$R_1(x) = \int_{x_j}^{x} (x-t)\, f''(t)\,dt$$

$$f(x) = f(x_j) + (x-x_j)\, f'(x_j) + \int_{x_j}^{x} (x-t)\, f''(t)\,dt$$

$$f(x_j+h) = f(x_j) + h\, f'(x_j) + \int_{x_j}^{x_j+h} (x_j+h-t)\, f''(t)\,dt$$

$$E_j = \int_{x_j}^{x_{j+1}} f(x)\,dx - h\,\frac{f(x_j)+f(x_{j+1})}{2}$$

$$= \int_{x_j}^{x_{j+1}} \left(f(x_j) + (x-x_j)\, f'(x_j) + \int_{x_j}^{x} (x-t)\, f''(t)\,dt \right) dx$$

$$- \frac{h}{2} f(x_j) - \frac{h}{2} \left(f(x_j) + h\, f'(x_j) + \int_{x_j}^{x_j+h} (x_j+h-t)\, f''(t)\,dt \right)$$

$$= \cancel{h f(x_j)} + \cancel{\frac{h^2}{2} f'(x_j)} \quad \int_{x_j}^{x_{j+1}} \int_{x_j}^{x} (x-t)\, f''(t)\,dt \; dx$$

$$- \cancel{\frac{h}{2} f(x_j)} - \frac{h}{2} \left(\cancel{f(x_j)} + \cancel{h\, f'(x_j)} + \int_{x_j}^{x_j+h} (x_j+h-t)\, f''(t)\,dt \right)$$

Since

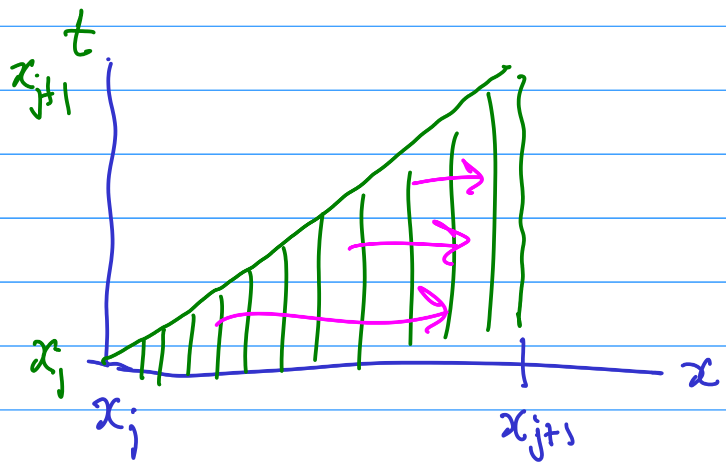
$$\int_{x_j}^{x_{j+1}} \left(f(x_j) + (x-x_j) f'(x_j) \right) = h f(x_j) + f'(x_j) \left. \frac{(x-x_j)^2}{2} \right|_{x_j}^{x_{j+1}}$$

$$= h f(x_j) + \frac{h^2}{2} f'(x_j)$$

Thus

$$E_j = \int_{x_j}^{x_{j+1}} \int_{x_j}^{x} (x-t) f''(t)\, dt\, dx - \frac{h}{2} \int_{x_j}^{x_j+h} (x_j+h-t) f''(t)\, dt$$

change order of integration.



$$= \int_{x_j}^{x_{j+1}} \int_{t}^{x_{j+1}} (x-t) f''(t)\, dx\, dt - \frac{h}{2} \int_{x_j}^{x_j+h} (x_j+h-t) f''(t)\, dt$$

const w.r.t to x

$$= \int_{x_j}^{x_{j+1}} \left[\int_t^{x_{j+1}} (x-t) f''(t)\, dx - \frac{h}{2} (x_j+h-t) f''(t) \right] dt$$

$$\int_t^{x_{j+1}} (x-t)\, dx = \frac{(x-t)^2}{2} \Big|_t^{x_{j+1}} = \frac{(x_{j+1}-t)^2}{2}$$

const x

$$E_j = \int_{x_j}^{x_{j+1}} \left[\frac{(x_{j+1}-t)^2}{2} f''(t) - \frac{h}{2} (x_j+h-t) f''(t) \right] dt$$

$$|E_j| \le \max_{x_j \le t \le x_{j+1}} |f''(t)| \int_{x_j}^{x_{j+1}} \left| \frac{(x_{j+1}-t)^2}{2} - \frac{h}{2} (x_{j+1}-t) \right| dt$$

now integrate this at home ... I'll post my result as well

$$|E_j| \le \frac{1}{12} h^3 \max_{x_j \le x \le x_{j+1}} |f''(x)|$$

Added after class: since $x_{j+1}-t \le x_{j+1}-x_j \le h$ then

$$\frac{(x_{j+1}-t)^2}{2} \le \frac{h}{2}(x_{j+1}-t)$$

and so what's inside the absolute value is always negative. Thus,

$$\int_{x_j}^{x_{j+1}} \left| \frac{(x_{j+1}-t)^2}{2} - \frac{h}{2}(x_{j+1}-t) \right| dt$$

$$= \int_{x_j}^{x_{j+1}} \left(\frac{h}{2}(x_{j+1}-t) - \frac{(x_{j+1}-t)^2}{2} \right) dt$$

$$= \left(-\frac{h}{2}\frac{(x_{j+1}-t)^2}{2} + \frac{(x_{j+1}-t)^3}{6} \right) \Bigg|_{x_j}^{x_{j+1}}$$

$$= \frac{h}{2}\frac{h^2}{2} - \frac{h^3}{6} = \frac{1}{12}h^3$$