$p_0(x) = \frac{1}{\sqrt{2}}$

$p_1(x) = \frac{x}{\sqrt{2/3}}$

$p_2(x) = \frac{x^2 - 1/3}{\sqrt{8/45}}$

$p_3(x) =$

$u_2 = x^2 - \frac{1}{3}$

$$\|u_2\|^2 = \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 \, dx = \int_{-1}^{1} (x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}) \, dx$$

$$= \left(\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{x}{9}\right)\Big|_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9}$$

$$= \frac{18-10}{45} = \frac{8}{45}$$

$$p_2 = \frac{u_2}{\|u_2\|} = \frac{x^2 - 1/3}{\sqrt{8/45}}$$

$u_3 = x^3 - (p_0, x^3)p_0 - (p_1, x^3)p_1 - (p_2, x^3)p_2$

$$(p_0, x^3) = \int_{-1}^{1} p_0(x)\, x^3 \, dx = \int_{-1}^{1} \frac{1}{\sqrt{2}} x^3 \, dx = 0$$

$$(p_1, x^3) = \int_{-1}^{1} p_1(x)\, x^3 \, dx = \int_{-1}^{1} \frac{x}{\sqrt{2/3}} x^3 \, dx = \frac{x^5}{5\sqrt{2/3}}\Big|_{-1}^{1} = \frac{2}{5\sqrt{2/3}}$$

$$(p_2, x^3) = \int_{-1}^{1} p_2(x) x^3 dx = \int_{-1}^{1} \frac{x^2 - 1/3}{\sqrt{8/45}} x^3 \, dx = \int_{-1}^{1} \frac{x^5 - x^3/3}{\sqrt{8/45}} dx = 0$$

$$u_3 = x^3 - \left(\frac{2}{5\sqrt{2/3}} \right) \frac{x}{\sqrt{2/3}} = x^3 - \frac{3}{5} x = x\left(x^2 - \frac{3}{5}\right)$$

roots are 0 and $\pm\sqrt{\frac{3}{5}}$

So $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{3/5}$

Now

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f\left(-\sqrt{\tfrac{3}{5}}\right) + w_2 f(0) + w_3 f\left(\sqrt{\tfrac{3}{5}}\right)$$

and I solve for the w's using

$$2 = \int_{-1}^{1} 1\,dx = w_1 + w_2 + w_3$$

$w_2 = 2 - w_1 - w_3$

$$0 = \int_{-1}^{1} x\,dx = w_1\left(-\sqrt{\tfrac{3}{5}}\right) + w_3\left(\sqrt{\tfrac{3}{5}}\right)$$

$-w_1 + w_3 = 0$

$w_1 + w_3 = \frac{10}{9}$

$$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = w_1 \frac{3}{5} + w_3 \frac{3}{5}$$

Thus $w_1 = w_3$ and $2w_3 = \frac{10}{9}$ so $w_3 = \frac{5}{9}$

..., also $w_1 = \frac{5}{9}$

$$w_2 = 2 - 2\frac{5}{9} = \frac{18}{9} - \frac{10}{9} = \frac{8}{9}$$

Thus, Gaussian Quadrature on 3 points is

$$\boxed{\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)}$$

This formula is exact when f is a polynomial of degree $\boxed{5}$ or less...

$2n-1$ where $n=3$

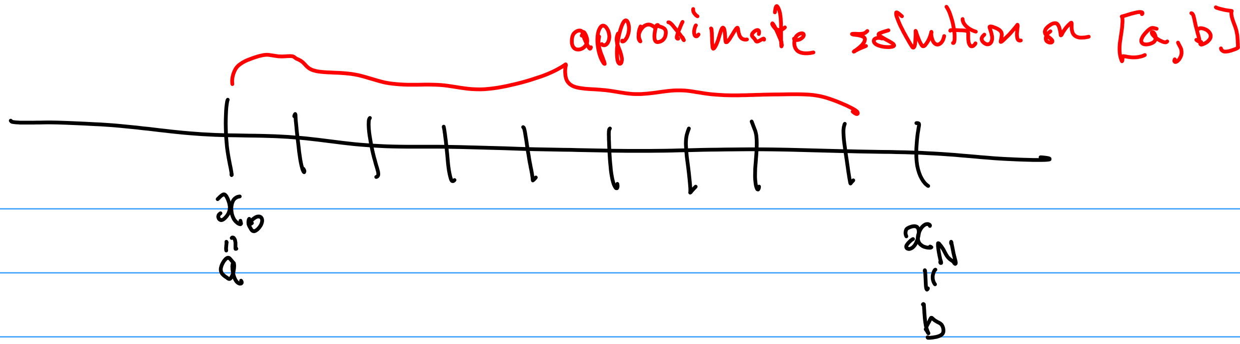
Step 32 on page 147

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \tfrac{5}{9} f(-\sqrt{3/5}) + \tfrac{8}{9} f(0) + \tfrac{5}{9} f(\sqrt{3/5})$$

Monday... finish Diff eq. and have question/Answer about the Final Review sheet posted over the weekend...

Differential equations.

Approximate solution to $\boxed{y' = f(x,y), \quad y(x_0) = y_0}$



$x_i = x_0 + hi$ where $h = \dfrac{b-a}{N}$

Taylor's theorem:

$$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + \cdots + \frac{h^n}{n!} y^{(n)}(x_n) + R_n$$

First order approximation.

$$\underbrace{y(x_{n+1})}_{\text{value at } x_{n+1}} \approx \underbrace{y(x_n) + h y'(x_n)}_{\text{values at } x_n}$$

Since

$$y' = f(x, y)$$

$$y'(x_n) = f(x_n, y(x_n))$$

$$y(x_{n+1}) \approx y(x_n) + h f(x_n, y(x_n))$$

Euler's method.

$$y_{n+1} \approx y_n + h f(x_n, y_n)$$ and compute for $n = 0, 1, \ldots, N-1$

Then $y_n \approx y(x_n)$

Approximation by Euler's method.

exact solution

Could write code using a loop

```
for n=0:N-1
    x = x0 + nh
    y = y + hf(x,y)
    println("y(", x+h, ") = ", y)
end
```

Maybe on Friday.

Second order approximation.

$$y(x_{n+1}) \approx y(x_n) + h\,y'(x_n) + \frac{h^2}{2} y''(x_n)$$

Since

$$y' = f(x,y) \qquad y'(x_n) = f(x_n, y(x_n))$$

$$y'' = \frac{d}{dx} f(x,y) = f_x(x,y) + f_y(x,y)\frac{dy}{dx} = f_x(x,y) + f_y(x,y)\,f(x,y)$$

$$y''(x_n) = f_x(x_n, y(x_n)) + f_y(x_n, y(x_n))\,f(x_n, y(x_n))$$

2nd order Taylor method   Let $y_n \approx y(x_n)$

and define

$$y_{n+1} = y_n + h f(x_n, y_n) + \frac{h^2}{2}\left(f_x(x_n, y_n) + f_y(x_n, y_n) f(x_n, y_n)\right)$$

```
for n=0:N-1
    x = x0 + nh
    y = y + h f(x, y) + h^2/2 (f_x(x,y) + f_y(x,y) f(x,y))
    println("y(", x+h, ") = ", y)
end
```

You can make higher order Taylor methods...,

Another idea use Quadrature to solve differential equations

$$y' = f(x, y)$$

Idea integrate both sides...

$$\int_{x_n}^{x_{n+1}} y'(x)\,dx = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

$y(x_{n+1}) - y(x_n) =$ Quadrature rule...

trapezoid rule

$$y(x_{n+1}) - y(x_n) \approx h \, \frac{f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))}{2}$$

Trying to find $y(x_{n+1})$ from $y(x_n)$ is difficult because $y(x_{n+1})$ appears here.