1. Find the range of solutions for the system, assuming maxiumu errors in the constants as shown:

$$x - y = 1.4 \pm 0.02$$
$$x + y = 3.8 \pm 0.03.$$

Augmented matrix: $[A\,|\,b] = \left[\begin{array}{cc|c} 1 & -1 & 1.4 \pm 0.02 \\ 1 & 1 & 3.8 \pm 0.03 \end{array}\right]$

Now do row operations keeping track of the errors

$$\left[\begin{array}{cc|c} 1 & -1 & 1.4 \pm 0.02 \\ 1 & 1 & 3.8 \pm 0.03 \end{array}\right] \qquad r_2 \leftarrow r_2 - r_1 \qquad \begin{array}{r} 3.8 \pm 0.02 \\ -\ 1.4 \pm 0.03 \\ \hline 2.4 \pm 0.05 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1.4 \pm 0.02 \\ 0 & 2 & 2.4 \pm 0.05 \end{array}\right] \qquad r_2 \leftarrow \tfrac{1}{2} r_2$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1.4 \pm 0.02 \\ 0 & 1 & 1.2 \pm 0.025 \end{array}\right] \qquad r_1 \leftarrow r_1 + r_2 \qquad \begin{array}{r} 1.4 \pm 0.02 \\ +\ 1.2 \pm 0.025 \\ \hline 2.6 \pm 0.045 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2.6 \pm 0.045 \\ 0 & 1 & 1.2 \pm 0.025 \end{array}\right] \qquad \begin{array}{l} x_1 = 2.6 \pm 0.045 \\ x_2 = 1.2 \pm 0.025 \end{array}$$

Solution

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.6 \pm 0.045 \\ 1.2 \pm 0.025 \end{bmatrix}.$$

2. How does partial pivoting contribute to a reduction of errors?

Partial pivoting chooses the entry of maximum norm for the pivots. This makes the denominators in the multipliers used for the elimination steps as large as possible. Doing so ensures all the α's in the row operations $r_i \leftarrow r_i - \alpha r_j$ are less than one. This means the errors in r_j are decreased as the are propagated forward by the Gaussian elimination algorithm.

3. Consider solving the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 5 & -3 & 0 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & 6 & -1 & 0 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & -3 & 5 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 3 \\ 2 \end{bmatrix}.$$

(i) Show that $x = (1, 1, 1, 1, 1)$ is the exact solution.

Just multiply Ax and check it's the same as b.

```
julia> A=[5 -3 0 0 0; -1 5 -1 0 0; 0 -1 6 -1 0; 0 0 -1 5 -1; 0 0 0 -3 5]
5×5 Matrix{Int64}:
  5  -3   0   0   0
 -1   5  -1   0   0
  0  -1   6  -1   0
  0   0  -1   5  -1
  0   0   0  -3   5

julia> x=[1,1,1,1,1]
5-element Vector{Int64}:
 1
 1
 1
 1
 1

julia> A*x
5-element Vector{Int64}:
 2
 3
 4
 3
 2
```

note that this is the same as b.

(ii) Use the `lu` function from Julia's `LinearAlgebra` library to find the LU factorization of A. Explain why L and U have so many zeros.

```
julia> using LinearAlgebra

julia> lu(A)
LU{Float64, Matrix{Float64}}
L factor:
5×5 Matrix{Float64}:
  1.0   0.0        0.0        0.0       0.0
 -0.2   1.0        0.0        0.0       0.0
  0.0  -0.227273   1.0        0.0       0.0
  0.0   0.0       -0.173228   1.0       0.0
  0.0   0.0        0.0       -0.621533  1.0
U factor:
5×5 Matrix{Float64}:
 5.0  -3.0   0.0       0.0       0.0
 0.0   4.4  -1.0       0.0       0.0
 0.0   0.0   5.77273  -1.0       0.0
 0.0   0.0   0.0       4.82677  -1.0
 0.0   0.0   0.0       0.0       4.37847
```

Since the original matrix already had lots of zeros in the upper and lower corners there was no need to eliminate them using row operations. As a result the zeros in the lower corner of L represent the fact that no elimination steps were performed between the corresponding rows when creating U. The zeros in the upper part of U were there in the beginning and remain after the elimination steps needed to produce the factorization.

(iii) Starting with $x_0 = (2,3,4,3,2)$ use 5 iterations of the Gauss-Seidel method to approximate the solution. Please include your code and the value of the n-th iteration x_n for $n = 1, \ldots, 5$ in your report.

The code from class for Gauss-Seidel was

```
julia> function gs(A,b,x)
           y=copy(x)
           for i=1:size(A)[1]
               t=b[i]
               for j=1:size(A)[2]
                   if i!=j
                       t-=A[i,j]*y[j]
                   end
               end
               t/=A[i,i]
               y[i]=t
           end
           return y
       end
gs (generic function with 1 method)
```

Calculate 5 iterations as

```
julia> b=[1,1,1,1,1]
5-element Vector{Int64}:
 1
 1
 1
 1
 1

julia> xn=[2.0,3.0,4.0,3.0,2.0]
5-element Vector{Float64}:
 2.0
 3.0
 4.0
 3.0
 2.0

julia> xn=gs(A,b,xn)
5-element Vector{Float64}:
 2.0
 1.4
 0.9
 0.78
 0.6679999999999999
```

1st

```
julia> xn=gs(A,b,xn)
5-element Vector{Float64}:
 1.0399999999999998
 0.588
 0.3946666666666667
 0.41253333333333336
 0.44752000000000003
```

2nd

```
julia> xn=gs(A,b,xn)
5-element Vector{Float64}:
 0.5528
 0.38949333333333336
 0.30033777777777776
 0.3495715555555556
 0.40974293333333334
```

3rd

```
julia> xn=gs(A,b,xn)
5-element Vector{Float64}:
 0.433696
 0.34680675555555557
 0.2827297185185185
 0.33849453037037036
 0.40309671822222215
```

4th

```
julia> xn=gs(A,b,xn)
5-element Vector{Float64}:
 0.4080840533333333
 0.33816275437037036
 0.2794428807901234
 0.33650791980246914
 0.40190475188148145
```

5th

Value of the 5th iteration

4. How is the condition number of a matrix defined and how is it used as a test for ill-conditioning?

The condition number of a matrix A is defined as

$$k(A) = \|A\| \|A^{-1}\|$$

This quantity appears in the backwards error analysis of the residual error in the approximation x^* of the exact solution to the equation $Ax = b$.

In particular let $b^* = Ax^*$. Then

$$\frac{\|x - x^*\|}{\|x\|} \leq k(A) \frac{\|b - b^*\|}{\|b\|}$$

Thus the condition number is used to bound the relative error in x^* by the relative error in the residual. The larger $k(A)$ is the poorer the conditioning.

Since relative error is related to significant digits, then if $k(A) \approx 10^m$ and computations are performed to k significant digits, the solution of $Ax=b$ will be accurate to $k-m$ decimal digits as plugging in the solution to $Ax=b$ can't distinguish more digits than that.

5. Consider the vector norms of a vector $x \in \mathbf{R}^n$ defined as

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{x \cdot x} \qquad \text{and} \qquad \|x\|_\infty = \max\{\, |x_i| : i = 1, \ldots, n \,\}.$$

Let $b = (-7, 8, -1, 2)$ and find $\|b\|_1$, $\|b\|_2$ and $\|b\|_\infty$.

$$\|b\|_1 = \sum_{i=1}^{4} |b_i| = |-7| + |8| + |-1| + |2| = 7 + 8 + 1 + 2 = 18.$$

$$\|b\|_2 = \sqrt{b \cdot b} = \sqrt{49 + 64 + 1 + 4} = \sqrt{118} \approx 10.86278\ldots$$

$$\|b\|_\infty = \max\{ |b_i| : i = 1, \ldots, 4 \} = \max\{ |-7|, |8|, |-1|, |2| \}$$

$$= \max\{ 7, 8, 1, 2 \} = 8.$$

6. [Extra Credit/Math 666] Is the ordering of the norms $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ from smallest to largest the same independent of the value of x? If so, explain why. If not provide a counter example.

In problem 5 the ordering of the norms was

$$\|b\|_\infty \leq \|b\|_2 \leq \|b\|_1$$

This ordering holds for any vector x as follows.

To see $\|x\|_\infty \leq \|x_2\|$ let m be an index such that

$$|x_m| = \max\{ |x_i| : i = 1, \ldots, n \} = \|x\|_\infty.$$

Then

$$|x_m|^2 \leq \sum_{i=1}^{n} |x_i|^2 = x \cdot x$$

implies $|x_m| \leq \sqrt{x \cdot x} = \|x\|_2$.

Consequently $\|x\|_\infty \leq \|x\|_2$.

To see $\|x\|_2 \leq \|x\|_1$ note that

$$\|x_1\|^2 = \left(\sum_{i=1}^{n} |x_i| \right) \left(\sum_{j=1}^{n} |x_j| \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j|$$

This is a sum of possitive terms that includes the terms when $i=j$.
Thus, setting

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

and noting that $\delta_{ij} \leq 1$ we obtain

$$\sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j| \delta_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j| = \|x_1\|^2.$$

Since $\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$, it follows that $\|x\|_2 \leq \|x\|_1$.

7. Consider the matrix

$$A = \begin{bmatrix} 4 & 8 & -4 & -10 \\ 8 & 3 & -10 & -1 \\ -1 & -7 & 6 & -9 \\ -8 & 7 & 1 & -3 \end{bmatrix}.$$

Find the matrix norms $\|A\|_\infty$ and $\|A\|_1$.

$$\|A\|_\infty = \max\left\{ \sum_{j=1}^{4} |a_{ij}| : i=1,\dots,4 \right\}$$

which is the largest among the absolute sums of the rows.

$$|4|+|8|+|-4|+|-10| = 26 \leftarrow \text{largest}$$

$$|8|+|3|+|-10|+|-1| = 22$$

$$|-1|+|-7|+|6|+|-9| = 23$$

$$|-8|+|7|+|1|+|-3| = 19$$

Therefore,

$$\|A\|_\infty = \max\{26, 22, 23, 19\} = 26$$

Also

$$\|A\|_1 = \max\left\{ \sum_{i=1}^{4} |a_{ij}| : j=1,\dots,4 \right\}$$

which is the largest among the absolute sums of the columns.

$$|4|+|8|+|-1|+|-8| = 21$$

$$|8|+|3|+|-7|+|7| = 25 \leftarrow \text{largest}$$

$$|-4|+|-10|+|6|+|1| = 21$$

$$|-10|+|-1|+|-9|+|-3| = 23$$

Therefore

$$\|A\|_1 = \max\{21, 25, 21, 23\} = 25$$