**1.** Let $f(x) = \sqrt{x}$ and $x_i = x_0 + hi$ where $x_0 = 1$ and $h = 1/2$. Consider the following table of finite differences:

| $x$ | $f(x)$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|
| 1 | 1.0000 | | | | | |
| | | 2247 | | | | |
| 1.5 | 1.2247 | | −352 | | | |
| | | 1895 | | 126 | | |
| 2 | 1.4142 | | −226 | | −59 | |
| | | 1669 | | 67 | | 28 |
| 2.5 | 1.5811 | | −159 | | −31 | |
| | | 1510 | | 36 | | |
| 3 | 1.7321 | | −123 | | | |
| | | 1387 | | | | |
| 3.5 | 1.8708 | | | | | |

**(i)** Use the approximation

$$f(x_i + \alpha h) \approx \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i$$

to show that

$$f(x) \approx p(x) \qquad \text{where} \qquad p(x) = \sum_{k=0}^{n} \binom{(x - x_i)/h}{k} \Delta^k f_i.$$

Setting $x_i + \alpha h = x$ and solving for $\alpha$ yields

$\alpha h = x - x_i$

$\alpha = (x - x_i)/h$

Therefore $f(x) \approx \sum_{k=0}^{n} \binom{(x-x_i)/h}{k} \Delta^k f_i$ which implies that $p(x) = \sum_{k=0}^{n} \binom{(x-x_i)/h}{k} \Delta^k f_i$.

**(ii)** Let $n = 3$ and $i = 1$ to find the unique polynomial $p$ of degree three such that

$$p(1.5) = 1.2247, \quad p(2) = 1.4142, \quad p(2.5) = 1.5811 \quad \text{and} \quad p(3) = 1.7321.$$

Please do not simplify or attempt to write $p$ in the standard basis.

In the table the points

$(1.5, 1.2247)$, $(2, 1.4142)$, $(2.5, 1.5811)$ and $(3, 1.7321)$

appear in the indicated rows

| $x$ | $f(x)$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|
| 1 | 1.0000 | | | | | |
| | | 2247 | | | | |
| 1.5 | 1.2247 | | −352 | | | |
| | | 1895 | | 126 | | |
| 2 | 1.4142 | | −226 | | −59 | |
| | | 1669 | | 67 | | 28 |
| 2.5 | 1.5811 | | −159 | | −31 | |
| | | 1510 | | 36 | | |
| 3 | 1.7321 | | −123 | | | |
| | | 1387 | | | | |
| 3.5 | 1.8708 | | | | | |

Therefore the differences needed are on the diagonal. The result upon setting $n=3$, $x_i = 1.5$ and $h = \frac{1}{2}$ is that

$$(x-x_i)/h = (x-1.5)/(1/2) = 2x-3$$

$$p(x) = \sum_{k=0}^{3} \binom{2x-3}{k} \Delta^k f_i .$$

$$= 1.2247 + 0.1895(2x-3) - \frac{0.0226}{2}(2x-3)(2x-4) + \frac{0.0067}{6}(2x-3)(2x-4)(2x-5)$$

We do not multiply the terms out since the request was not to simplify the answer.

**(iii)** Are the third-order differences constant to within the expected rounding error? What does this tell you about the quality of the approximation $f(x) \approx p(x)$?

The expected rounding error is given by the table on page 81 in the text.

| *Tabular error* | *Differences* | | | | | |
|---|---|---|---|---|---|---|
| | *1st.* | *2nd.* | *3rd.* | *4th.* | *5th.* | *6th.* |
| $+\frac{1}{2}$ | | | | | | |
| | $-1$ | | | | | |
| $-\frac{1}{2}$ | | $+2$ | | | | |
| | $+1$ | | $-4$ | | | |
| $+\frac{1}{2}$ | | $-2$ | | $+8$ | | |
| | $-1$ | | $+4$ | | $-16$ | |
| $-\frac{1}{2}$ | | $+2$ | | $-8$ | | $+32$ |
| | $+1$ | | $-4$ | | $+16$ | |
| $+\frac{1}{2}$ | | $-2$ | | $+8$ | | |
| | $-1$ | | $+4$ | | | |
| $-\frac{1}{2}$ | | $+2$ | | | | |
| | $+1$ | | | | | |
| $+\frac{1}{2}$ | | | | | | |

Therefore, the third differences should be within $\pm 4$ of each other if the resulting interpolation is to be good to the 5 significant digits that appear in the table.

Since the values of $\Delta^3 f_i$ given as 126, 67 and 36 do not meet this requirement, then the quality of the interpolation may not be good to 5 significant digits.

**(iv)** Evaluate $p(2.25)$ and compute $\left|\sqrt{2.25} - p(2.25)\right|$ to find the error.

```
julia> p(x)=1.2247+(2*x-3)*(0.1895+(2*x-4)*(-0.0226/2+(2*x-5)*0.0067/6))
p (generic function with 1 method)

julia> p(2.25)
1.5000562499999999

julia> abs(sqrt(2.25)-p(2.25))
5.6249999999868905e-5
```

Note that its also possible to use the code developed in class to compute the answer to this problem.

```
f(x)=sqrt(x)
xs=1:0.5:3.5
N=length(xs)
M=zeros(N,N)
for i=1:N
    M[i,1]=f(xs[i])
end
for j=2:N
    for i=1:N-j+1
        M[i,j]=M[i+1,j-1]-M[i,j-1]
    end
end

function p(alpha,d)
    b=1.0
    s=0
    for j=1:d+1
        s+=b*M[2,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```

```
julia> g(x)=p(2*x-3,3)
g (generic function with 1 method)

julia> g(2.25)
1.5000859908221935

julia> abs(sqrt(2.25)-g(2.25))
8.599082219351573e-5
```

The rounding is different since the computer made all computations using 64-bit floating point, which is good to about 15 decimal digits.
The error is the same order of magnitude

**2.** The polynomial interpolation theorem states $f(t) = p(t) + E(t)$ where

$$E(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (t - x_{i+k})$$

for some $\xi$ between $\min\{t, x_i, \ldots, x_{i+n}\}$ and $\max\{t, x_i, \ldots, x_{i+n}\}$.

**(i)** Let $f(x) = \sqrt{x}$ and let $B = \max\{\, |f^{(4)}(\xi)| : \xi \in [1.5, 3]\,\}$. Find $B$.

Since

$$f'(x) = \frac{1}{2} x^{-1/2}, \quad f''(x) = -\frac{1}{4} x^{-3/2}, \quad f'''(x) = \frac{3}{8} x^{-5/2}, \quad f^{(4)}(x) = \frac{-15}{16} x^{-7/2}$$

then

$$|f^{(4)}(x)| = \frac{15}{16} x^{-7/2}$$

This is a decreasing function on $[1.5, 3]$ therefore

$$B = \max\left\{ \frac{15}{16} \xi^{-7/2} : \xi \in [1.5, 3] \right\} = \frac{15}{16} (1.5)^{-7/2}$$

$$\approx 0.2268\ldots$$

```
julia> 15/16*(1.5)^(-7/2)
0.22680460581325723
```

**(ii)** Consider the interpolating polynomial $p$ of degree three such that

$$p(1.5) = 1.2247, \quad p(2) = 1.4142, \quad p(2.5) = 1.5811 \quad \text{and} \quad p(3) = 1.7321.$$

For this polynomial it follows for $t \in [1.5, 3]$ that

$$|E(t)| \le \frac{B}{(n+1)!}\Big|(t-1.5)(t-2)(t-2.5)(t-3)\Big|.$$

Evaluate this theoretical upper bound on the error $|E(2.25)|$.

Since $n=3$ then

$$|E(2.25)| \le \frac{0.2268\ldots}{4!}\big|(2.25-1.5)(2.25-2)(2.25-2.5)(2.25-3)\big|$$

$$\le \frac{0.2268\ldots}{4!}\big|(0.75)(0.25)(-0.25)(-0.75)\big|$$

$$\approx 0.00033223\ldots$$

```
julia> B/factorial(4)*0.75*0.25*0.25*0.75
0.00033223333092967635
```

**(iii)** How many times bigger is the theoretical upper bound on $|E(2.25)|$ compared to the actual error $\left|\sqrt{2.25} - p(2.25)\right|$ from 1(iv) of the previous problem? Is it possible the actual error could be bigger than theoretical upper bound?

```
julia> Etheory=B/factorial(4)*0.75*0.25*0.25*0.75
0.00033223333092967635

julia> Eactual=abs(sqrt(2.25)-p(2.25))
5.6249999999868905e-5

julia> Etheory/Eactual
5.906369943067339
```

Alternatively, if you used the difference table code from class one obtains

```
julia> Etheory=B/factorial(4)*0.75*0.25*0.25*0.75
0.00033223309296763 5

julia> Eactual=abs(sqrt(2.25)-g(2.25))
8.599082219351573e-5

julia> Etheory/Eactual
3.863590332339166
```

Therefore, the theoretical bound is 3 to 6 times greater than the actual error, depending on how the polynomial was obtained.

It could happen that rounding errors make the actual error greater than the theoretical bound, because rounding wasn't taken into the theory. On the other hand, this is unlikely, so, in general, the theoretical bound should be greater, especially since B is taken as the maximum.

**(iv)** [Extra Credit and Math 666] Compute the actual value of E(2.25) and solve to find $\xi$ in the polynomial interpolation theorem. Verify that $\xi \in [1.5, 3]$.

For the extra credit, we use the polynomial found using the code from class since that one has less rounding error.

We solve for $\xi$ such that

$$\sqrt{2.25} - g(2.25) = \frac{-\frac{15}{16}\xi^{-7/2}}{4!}(0.75)(0.25)(-0.25)(-0.75)$$

Thus,

$$\xi^{-7/2} = \frac{\sqrt{2.25} - g(2.25)}{(0.75)(0.25)(-0.25)(-0.75)} \cdot \frac{4!}{(-15/16)}$$

$$\xi = \left(\frac{\sqrt{2.25} - g(2.25)}{(0.75)^2(0.25)^2} \cdot \frac{24 \cdot 16}{-15}\right)^{-2/7}$$

$$= \left(\frac{(0.75)^2(0.25)^2}{\sqrt{2.25} - g(2.25)} \cdot \frac{-15}{24 \cdot 16}\right)^{2/7}$$

$$\approx 2.2070\ldots$$

```
julia> ((0.75)^2*(0.25)^2/(sqrt(2.25)-g(2.25))*(-15/(24*16)))^(2/7)
2.207003355900984
```

Therefore, $\xi$ is in the interval $[1.5, 3]$.