Taylor's Theorem. Suppose $f:\mathbb{R}\to\mathbb{R}$ with $n+1$ continuous derivatives, then

$$f(x)=\frac{(x-x_0)^0}{0!}f(x_0)+\frac{(x-x_0)}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+R_n(x)$$

where $R_n(x)=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c)$ for some c between x and x_0

Taylor's Theorem with integral remainder
n^{th} order

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)$$

positive term if $x>x_0$

where $R_n(x)=\int_{x_0}^{x}\boxed{\frac{(x-t)^n}{n!}}f^{(n+1)}(t)\,dt$

for definiteness assume $x>x_0$.

Let $m=\min\{f^{(n+1)}(t): t\in[x_0,x]\}$ let $M=\max\{f^{(n+1)}(t): t\in[x_0,x]\}$

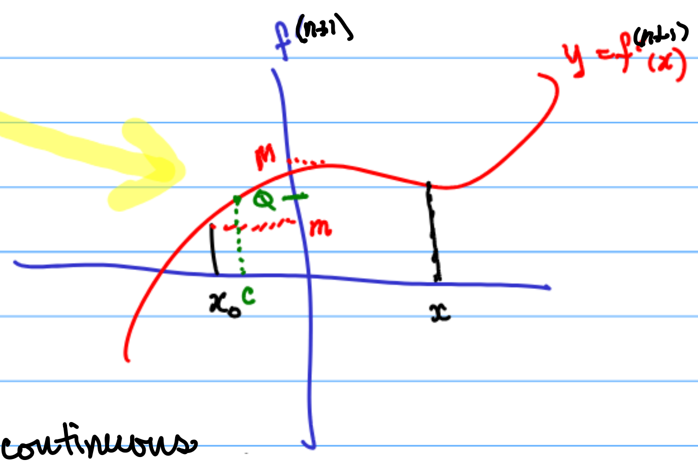
$$m\le f^{(n+1)}(t)\le M \qquad \text{for } t\in[x_0,x]$$

$$\frac{(x-t)^n}{n!}m\le\frac{(x-t)^n}{n!}f^{(n+1)}(t)\le\frac{(x-t)^n}{n!}M$$

$$m\int_{x_0}^{x}\frac{(x-t)^n}{n!}dt\le\int_{x_0}^{x}\frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt\le M\int_{x_0}^{x}\frac{(x-t)^n}{n!}dt$$

$$m \le \frac{\displaystyle\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}{\displaystyle\int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt} \le M$$

Q



Because $f^{(n+1)}(t)$ is continuous

$$\frac{\displaystyle\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}{\displaystyle\int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt} = f^{(n+1)}(c) \quad \text{for some } c \text{ between } x_0 \text{ and } x.$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt = \int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt \; f^{(n+1)}(c)$$

$$\int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt = -\frac{(x-t)^{n+1}}{(n+1)!}\Bigg|_{x_0}^{x} = \frac{(x-x_0)^{n+1}}{(n+1)!}$$

Therefore

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad \text{for some } c \text{ between } x_0 \text{ and } x.$$

Idea: iterate the approximation:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

where $\boxed{x_0 \text{ is the starting value}}$ of the iteration (hopefully close to $\sqrt{2}$ so that the first approx is not too bad).

This is like...

$$x_{n+1} = g(x_n) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)}$$

we are looking for the root $x = \alpha$ where $f(\alpha) = 0$.

Note $g(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)} = \alpha - \frac{0}{f'(\alpha)} = \alpha$

assuming $f'(\alpha) \neq 0$

In general lets consider a scheme given by

$$x_{n+1} = g(x_n) \quad \text{where} \quad g(\alpha) = \alpha$$

α is a fixed point of g

Error

$e_n = x_n - \alpha$

Estimate e_{n+1} in terms of e_n.

mean value theorem
or 0^{th} order Taylor...

$$e_{n+1} = x_{n+1} - \alpha = g(x_n) - g(\alpha) = g'(c_n)(x_n - \alpha) = g'(c_n)\, e_n$$

Thus $|e_{n+1}| = |g'(c_n)||e_n|$ for some c_n between x_n and α.

If $|g'(c_n)| < 1$ then the e_{n+1} error is smaller than the e_n error...

Suppose that $|g'(\alpha)| < 1$ and g' is continuous.
Then there is $\delta > 0$ and a $0 < \gamma < 1$ such that

$$|g'(x)| \le \gamma \quad \text{for} \quad |x - \alpha| < \delta.$$

Let x_0 be an approximation of α where $|x_0 - \alpha| < \delta$.

$|e_1| = |g'(c_0)\, e_0| \le \gamma |e_0| \le \gamma\delta$ since c_0 is between x_0 and α
then $|c_0 - \alpha| < \delta$
so $|g'(c_0)| \le \gamma$

$|e_2| = |g'(c_1)\, e_1| \le \gamma |e_1| \le \gamma^2 \delta$ since c_1 is between x_1 and α
and $|x_1 - \alpha| \le \gamma\delta < \delta$ then $|g'(c_1)| \le \gamma$

⋮ by induction

$|e_n| \le \gamma^n \delta$ for all n.

Since $0<\gamma<1$ then $|e_n| \le \gamma^n \delta \to 0$ as $n \to \infty$.

In other words $x_n \to \alpha$ as $n \to \infty$.

Theorem: Suppose $x_{n+1} = g(x_n)$ with $g(\alpha)=\alpha$ and g' exists and is continuous.
If $|g'(\alpha)|<1$ there is a $\delta>0$ such that $|x_0-\alpha|<\delta$ implies $x_n \to \alpha$ as $n \to \infty$.

Note we have an estimate of the rate of convergence.

$|e_n| \le \gamma^n \delta$ so as $n \to \infty$, $|e_n|$ exponentially decreases...

Recall this came from the estimate $|e_{n+1}| \le \gamma |e_n|$

this is a linear relationship between the error e_n and the next error e_{n+1}

Back to Newton's method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{(f'(\alpha))^2} = 0$$ this the best case of being less than 1.