```
julia> xn=g(xn)
```

5

```
1.414285714285714285714285714285714285714285714285714285714285714285714
428571428571428
```

```
julia> xn=g(xn)
```

9

```
1.414213564213564213564213564213564213564213564213564213564213564213564213
421356421356421
```

```
julia> xn=g(xn)
```

18

```
1.414213562373095049999289578902863939341670069641500982111155061514737041496
```

```
julia> xn=g(xn)
```

36

```
1.414213562373095048801688724209698079076755050549970818145556054685945434484 37
```

```
julia> xn=g(xn)
1.414213562373095048801688724209698078569671875376948073176679737990732478553006

julia> xn=g(xn)
1.414213562373095048801688724209698078569671875376948073176679737990732478462102

julia> xn=g(xn)
1.414213562373095048801688724209698078569671875376948073176679737990732478462102
```

General Methods of the form

$$x_{n+1} = g(x_n) \qquad \text{where} \qquad g(\xi) = \xi$$

We know this iteration converges if there is $\delta > 0$ and $\gamma \in [0,1)$ such that $|g'(x)| \le \gamma$ for $x \in [\xi-\delta, \xi+\delta]$ and $x_0 \in [\xi-\delta, \xi+\delta]$.

Newton...

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Simpler

$$g(x) = x - \lambda f(x)$$

Question: for what values of λ does the iteration $x_{n+1} = g(x_n)$ converge?

We know if there is $\delta > 0$ and $0 < \theta < 1$ where

$$|g'(x)| \le \theta \quad \text{for } x \in [\xi-\delta, \xi+\delta] \quad \text{and} \quad x_0 \in [\xi-\delta, \xi+\delta]$$

then the iteration converges.

$$g'(x) = 1 - \lambda \underbrace{f'(x)}_{\text{bound the derivative from below and above.}}$$

Suppose $\underbrace{m \le f'(x) \le M}$ for $x \in [\xi-\delta, \xi+\delta]$

note if m and M are both positive this condition guarantees that $f'(x) \ne 0$.

note if m and M are both negative this condition guarantees that $f'(x) \ne 0$.

If they are both negative one can multiply by -1 or redo the argument slightly modified.

$$m \le f'(x) \le M$$

$$\underbrace{1-\lambda M}_{-\theta} \le 1-\lambda f'(x) \le \underbrace{1-\lambda m}_{\theta}$$

other choices could also lead to $|g'(x)| \le \theta$, but this way is somehow the best ...

Thus

$$\begin{aligned} 1-\lambda M &= -\theta \\ 1-\lambda m &= \theta \\ \hline 1-\lambda M + 1-\lambda m &= 0 \end{aligned}$$

$$\lambda(M+m) = 2$$

$$\lambda = \frac{2}{M+m} \qquad \underbrace{\theta = 1-\lambda m}_{\text{this implies } \theta \in [0,1).} = 1-\frac{2}{M+m}m = \frac{M+m-2m}{M+m} = \frac{M-m}{M+m}$$

Which means there is a choice of λ so that $x_{n+1}=g(x_n)$ converges provided x_0 is close enough to the root ξ. And $f'(\xi) \neq 0$.

Note, I don't know what λ is because I don't know what m and M are because I'm not really going to find f'.

One expects linear convergence with the relaxation method which means at each iteration a fixed number of digits more are made correct each time.

Other ways to avoid computing f' each iteration...

Hybrid Newton/Relaxation...

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \text{Newton step}$$

$$x_2 = x_1 - \frac{1}{f'(x_0)} f(x_1) \qquad \text{Relaxation } \lambda = \frac{1}{f'(x_0)}$$

$$x_3 = x_2 - \frac{1}{f'(x_0)} f(x_2) \qquad \text{Relaxation } \lambda = \frac{1}{f'(x_0)}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \qquad \text{Newton step}$$

$$x_5 = x_4 - \frac{1}{f'(x_3)} f(x_4) \qquad \text{Relaxation } \lambda = \frac{1}{f'(x_3)}$$

$$x_6 = x_5 - \frac{1}{f'(x_3)} f(x_5) \qquad \text{Relaxation } \lambda = \frac{1}{f'(x_3)}$$

<u>Another Idea:</u> Multistep method... (one step is like Newton w/ $\ell = 0$, two step is $\ell = 1$)

$$x_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-\ell}) \quad \text{for } n = \ell, \ell+1, \dots$$

where $x_0, x_1, \dots, x_\ell$ are different but close approximations of the root ξ.

Secant Method. $\qquad f'(x_n) \approx \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

$$\underbrace{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}_{\text{Newton}} \approx \underbrace{x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}}_{\text{Secant}}$$

Thus
$$g(x_n, x_{n-1}) = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

and Secant method is

$$x_{n+1} = g(x_n, x_{n-1})$$