Definition 1.7 *Suppose that $\xi = \lim_{k\to\infty} x_k$. We say that the sequence (x_k) converges to ξ* **with at least order** $q > 1$*, if there exist a sequence (ε_k) of positive real numbers converging to 0, and $\mu > 0$, such that*

$$|x_k - \xi| \le \varepsilon_k\,, \quad k = 0, 1, 2, \ldots, \qquad \text{and} \qquad \lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \mu\,. \qquad (1.21)$$

If (1.21) holds with $\varepsilon_k = |x_k - \xi|$ for $k = 0, 1, 2, \ldots$, then the sequence (x_k) is said to converge to ξ **with order** q. *In particular, if $q = 2$, then we say that the sequence (x_k) converges to ξ* **quadratically**.

We showed $|e_{n+1}| \le M|e_n|^2$ where $e_n = x_n - \xi$ the first day...

Suppose $\displaystyle\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \mu\,.$

This means for every $\varepsilon > 0$ there is K large enough

such that $\displaystyle\left|\frac{\varepsilon_{k+1}}{\varepsilon_k^q} - \mu\right| < \varepsilon$ for all $k \geq K$.

Thus

$$-\varepsilon < \frac{\varepsilon_{k+1}}{\varepsilon_k^q} - \mu < \varepsilon$$

or

$$-\varepsilon\,\varepsilon_k^q < \varepsilon_{k+1} - \mu\,\varepsilon_k^q < \varepsilon\,\varepsilon_k^q$$

$$(\mu - \varepsilon)\,\varepsilon_k^q < \boxed{\varepsilon_{k+1} < (\mu + \varepsilon)\,\varepsilon_k^q}$$

Since $\varepsilon_k > 0$ this is similar to $|e_{n+1}| \le M|e_n|^2$ with $q = 2$

and $M = \mu + \varepsilon$.

and $\varepsilon_k = |e_{k+1}|$

The connection the other way is a little more complicated

If $|e_{n+1}| \leq M|e_n|^2$ can I relate it to $\displaystyle\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \mu$.

where does this inequality go?

here

$|e_k| = |x_k - \xi| \leq \varepsilon_k$;

Theorem 1.8 (Convergence of Newton's method) *Suppose that f is a continuous real-valued function with continuous second derivative f'', defined on the closed interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, such that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Suppose further that there exists a positive constant A such that*

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall x, y \in I_\delta .$$

More careful only need this to hold on I_δ

If $|\xi - x_0| \leq h$, where h is the smaller of δ and $1/A$, then the sequence (x_k) defined by Newton's method (1.20) converges quadratically to ξ.

$f'(x_n)$

$$e_{n+1} = \frac{e_n^2}{2} \frac{f''(c_n)}{f'(x_n)}$$

assume this quotient is bounded by M

?- holds everywhere before?

for $\dfrac{|f''(x)|}{|f'(y)|}$ to be bounded, we need $f'(y)$ away from zero and $f''(x)$ away from infinity.

If $f''(x)$ is continuous, then it has a min and max on any closed and bounded interval $I_\delta = [\xi - \delta, \xi + \delta]$.

If $f'(\xi) \neq 0$ and f' is continuous then.

$$\alpha = f'(\xi) \quad \text{and} \quad \varepsilon = \frac{|\alpha|}{2}$$

then continuity implies there is $\delta > 0$ such that

$$|f'(y) - \alpha| < \varepsilon \quad \text{for every} \quad |y - \xi| < \delta$$

$$-\varepsilon < f'(y) - \alpha < \varepsilon$$

$$-\frac{|\alpha|}{2} < f'(y) - \alpha < \frac{|\alpha|}{2}$$

$$\alpha - \frac{|\alpha|}{2} < f'(y) < \alpha + \frac{|\alpha|}{2}$$

Need to show that $0 \notin \left[\alpha - \frac{|\alpha|}{2}, \alpha + \frac{|\alpha|}{2}\right]$.

Check if $\alpha > 0$ then $\left[\alpha - \frac{|\alpha|}{2}, \alpha + \frac{|\alpha|}{2}\right] = \left[\frac{\alpha}{2}, \frac{3\alpha}{2}\right]$

$\alpha < 0$ then $\left[\alpha - \frac{|\alpha|}{2}, \alpha + \frac{|\alpha|}{2}\right] = \left[\frac{3\alpha}{2}, \frac{\alpha}{2}\right]$

0 is not in either interval.

Let y_{min} be such that $|f'(y_{min})|$ is the smallest on I_δ. Then

$|f'(y)| \geq |f'(y_m)|$ implies $\dfrac{1}{|f'(y)|} \leq \dfrac{1}{|f'(y_{min})|} < \infty$.

In summary. Take $\delta > 0$ corresponding to $\varepsilon = \frac{|f'(\xi)|}{2}$ in the definition of continuity for f'. Then

$$\left|\frac{f''(x)}{f'(y)}\right| < \frac{\max\{|f''(x)| : x \in I_\delta\}}{\min\{|f'(y)| : y \in I_\delta\}} = A < \infty.$$

Theorem 1.8 (Convergence of Newton's method) *Suppose that f is a continuous real-valued function with continuous second derivative f'', defined on the closed interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, such that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Suppose further that there exists a positive constant A such that*

$$\frac{|f''(x)|}{|f'(y)|} \leq A \qquad \forall\, x, y \in I_\delta\,.$$

If $|\xi - x_0| \leq h$, where h is the smaller of δ and $1/A$, then the sequence (x_k) defined by Newton's method (1.20) converges quadratically to ξ.

• Change is identifying the interval I_δ and showing that x_k stay in this interval for all k.

Proof: Let $h = \min\left(\delta, \frac{1}{A}\right)$. $e_k = x_k - \xi$.

Then

$$e_{k+1} = \underbrace{x_{k+1}}_{\text{Newton step}} - \xi = x_k - \frac{f(x_k)}{f'(x_k)} - \xi = e_k - \frac{f(x_k)}{f'(x_k)}$$

By Taylor's theorem

divide by this

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{(\xi - x_k)^2}{2} f''(c_k)$$

for some c_k between x_k and ξ.

Thus
$$\frac{f(x_k)}{f'(x_k)} + (\xi - x_k) + \frac{(\xi - x_k)^2}{2} \frac{f''(c_k)}{f'(x_k)} = 0.$$

solve for this

or

$$\frac{f(x_k)}{f'(x_k)} = \underbrace{-(\xi - x_k)}_{e_k} - \frac{(\overbrace{\xi - x_k}^{-e_k})^2}{2} \frac{f''(c_k)}{f'(x_k)}$$

substitute

$$\frac{f(x_k)}{f'(x_k)} = e_k - \frac{e_k^2}{2} \frac{f''(c_k)}{f'(x_k)}$$

Then

$$e_{k+1} = \cancel{e_k} - \left(\cancel{e_k} - \frac{e_k^2}{2} \frac{f''(c_k)}{f'(x_k)}\right)$$

So $$e_{k+1} = \frac{e_k^2}{2} \frac{f''(c_k)}{f'(x_k)}$$

Since $|\xi - x_0| \leq h$, *where h is the smaller of δ and $1/A$,*

then $|e_0| \leq \min(\delta, 1/A)$

also c_0 between x_0 and ξ.

so $|c_0 - \xi| \leq |x_0 - \xi| \leq h$

Therefore $\left|\frac{f''(c_0)}{f'(x_0)}\right| \leq A$ so

$$|e_1| \leq \frac{A}{2}|e_0|^2 \leq \frac{A|e_0||e_0|}{2} \leq \frac{1}{2}|e_0|.$$

By induction..

Suppose $|e_k| \leq h$. Claim $|e_{k+1}| \leq h$.

Since c_k is be x_k and ξ

then $|c_k - \xi| \leq |x_k - \xi| = |e_k| \leq h$

Thus $\left|\frac{f''(c_k)}{f'(x_k)}\right| \leq A$ so

$$|e_{n+1}| \leq \frac{A}{2}|e_n|^2 \leq \frac{A|e_n||e_n|}{2} \leq \frac{1}{2}|e_n| \leq \frac{h}{2} \leq h$$

It follows by induction that

$$|e_n| \leq h \text{ for all } n$$

and

$$|e_{k+1}| \leq \frac{1}{2}|e_k| \quad \text{for all } k.$$

Thus. $|e_k| \leq \frac{1}{2^k}|e_0| \to 0$ as $k \to \infty$.

and it converges...

and $\left|\frac{f''(c_k)}{f'(x_k)}\right| \leq A$ for all k.

Thus

$$|e_{k+1}| \leq \frac{A}{2}|e_k|^2 \quad \text{for all } k.$$

This is the quadratic convergence.