Chapter 1: Focus solving a single non-linear equation

Newton, Secant, etc. . iterative methods

Chapter 2 Focus solving multiple linear equations

$Ax=b$ ... Gaussian elimination

$A=LU$ Factorization..

not iterative methods, called direct methods

Note, there are also iterative methods for solving $Ax=b$ (see Chapter 4)

Start with determinants...

Definition 2.1 *The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$. A matrix of size $n \times n$ will be called a square matrix of order n, or simply a matrix of* **order** *n. The* **determinant** *of a square matrix $A \in \mathbb{R}^{n\times n}$ is the real number $\det(A)$ defined as follows:*

$$\det(A) = \sum_{\text{perm}} \operatorname{sign}(\nu_1, \nu_2, \ldots, \nu_n) a_{1\nu_1} a_{2\nu_2} \ldots a_{n\nu_n} .$$

The summation is over all $n!$ permutations $(\nu_1, \nu_2, \ldots, \nu_n)$ of the integers $1, 2, \ldots, n$, and $\operatorname{sign}(\nu_1, \nu_2, \ldots, \nu_n) = +1$ or -1 depending on whether the n-tuple $(\nu_1, \nu_2, \ldots, \nu_n)$ is an even or odd permutation of $(1, 2, \ldots, n)$, respectively. An even (odd) permutation is obtained by an even (odd) number of exchanges of two adjacent elements in the array $(1, 2, \ldots, n)$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be **nonsingular** *when its determinant $\det(A)$ is nonzero.*

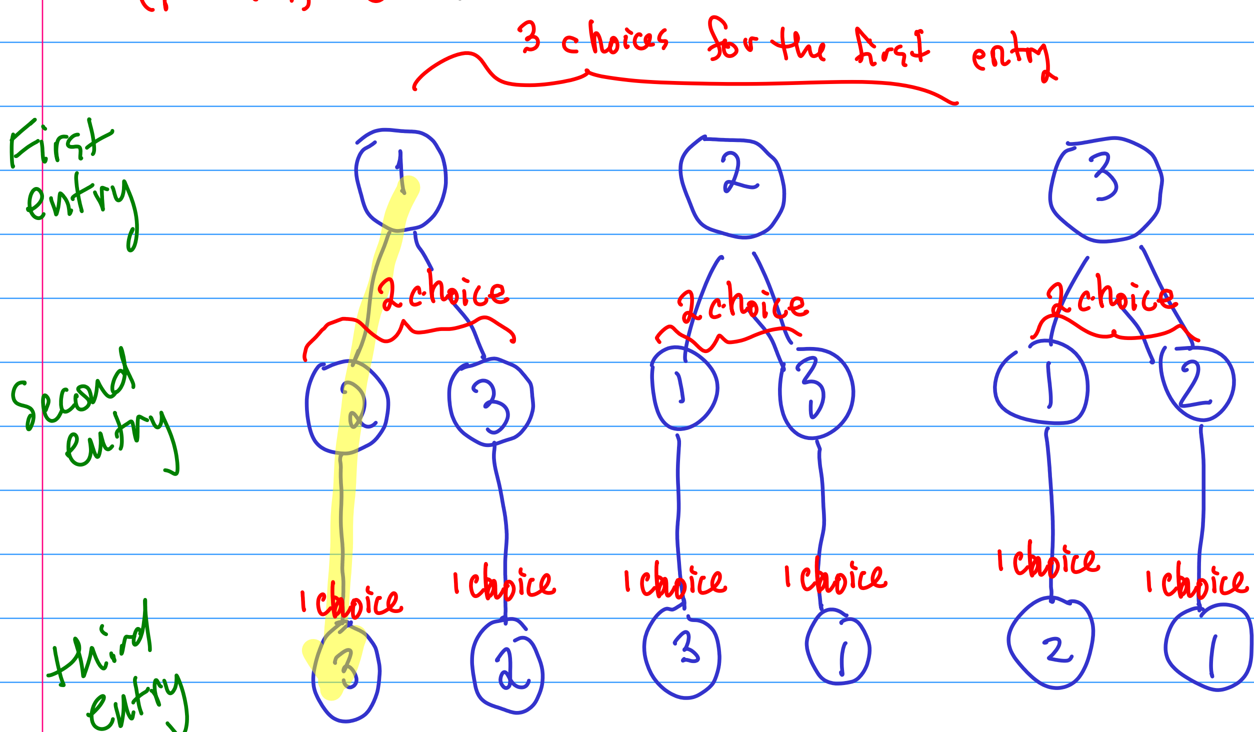
$\text{perm}(n) = \{ (v_1, v_2, \ldots, v_n) : (v_1, v_2, \ldots, v_n) \text{ is a permutation of } (1, 2, \ldots, n) \}$

permutations of n things

$\text{card}(\text{perm}(n)) = n!$

perm(3) find vectors of length three that are permutations of (1,2,3)

$\text{card}(\text{perm}(3)) = 6 = 3\cdot2\cdot1$



Which are even and which odd?

	start		swaps	sign(v)
(1, 2, 3)	(1,2,3)	no need to exchange anything	0 = even	1
(1, 3, 2)	(1,2,3) → (1,3,2)		1 = odd	-1
(2, 1, 3)	(1,2,3) → (2,1,3)		1 = odd	-1
(2, 3, 1)	(1,2,3) $\xrightarrow{1}$ (2,1,3) $\xrightarrow{2}$ (2,3,1)		2 = even	1
(3, 1, 2)	(1,2,3) $\xrightarrow{1}$ (3,2,1) $\xrightarrow{2}$ (3,1,2)		2 = even	1
(3, 2, 1)	(1,2,3) $\xrightarrow{1}$ (3,2,1)		1 = odd	-1

Recall recursive definition.

Let $\mathrm{Cof}(a_{ij})$ be the $(n-1)\times(n-1)$ determinant of the matrix A after crossing out the i^{th} row and j^{th} column

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\,\mathrm{Cof}(a_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\,\mathrm{Cof}(a_{ij})$$

$n\times n$ determinant

for all j the same answer

expansion along a column

for any i the same answer.

expansion along a row.

$$\det [a_{11}] = a_{11}$$

1×1 determinant

Another way to compute the determinant

$$A = P\,L\,U$$

upper triangular

lower triangular

permutation matrix.

multiplying a matrix on the right by a permutation matrix permutes

2 (a) the rows

5 (b) the columns

of that matrix.

$$\det A = \det P \det L \det U$$

± 1 depending whether the perm is even or odd.

prod on diag since lower triangular

prod on diag since upper triangular

Idea solving $Ax = b$...

Use Cramer's rule. If $\det A \neq 0$ then the unique solution is

$$x_i = \frac{\det A_i(b)}{\det A}$$

where $A_i(b)$ is the matrix A with i^{th} column replaced by b.

Theoretically nice but not very efficient or accurate numerically.

Find the inverse A^{-1} using cofactors...

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

where $A_{ij} = (-1)^{i+j}\mathrm{Cof}(a_{ij})$

even worse for efficiency and accuracy.

Of course the efficiency of the above formula depend on how quickly we can compute determinants.

3 ways to compute determinants

① by the definition

$$\det(A) = \sum_{\text{perm}} \operatorname{sign}(\nu_1, \nu_2, \ldots, \nu_n) a_{1\nu_1} a_{2\nu_2} \ldots a_{n\nu_n}$$

card(perm(n)) = n! so there are lots of terms here...

② Recursively

$$\underset{n \times n}{\det A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \operatorname{Cof}(a_{ij})$$

n determinants of size n-1 x n-1

n-1 determinants of size n-2 x n-2

⋮

1

$\underbrace{\qquad}$

n! terms...

③ Using Gauss elimination A = PLU

or solve $Ax = b$ with this factorization not using any determinants

$$\det A = \det P \det L \det U.$$

more efficient

n variables to eliminate

n equations with each variable

$\underline{n \text{ terms in each equation}}$

n^3 operations.

$A = PLU$

$P^{-1} = P^T$

since P is permutation

$Ax = b$

$PLUx = b$

$LUx = P^{-1}b$

$y = Ux$

$Ly = P^{-1}b$

Solve $Ly = P^{-1}b$
for y

then solve $Ux = y$
for x

Since L and U are triangular
than solving these equations can
be done with substitution.