For $p \in [1,\infty)$ define.

$$\|r\|_p = \sqrt[p]{\sum_{i=1}^{n} |r_i|^p}$$

$$\|r\|_\infty = \max\{|r_1|, |r_2|, \dots, |r_n|\}$$

<u>Claim</u> $\lim_{p\to\infty} \|r\|_p = \|r\|_\infty$ (see this later).

✓ ❶ $\|v\| = 0$ *if, and only if,* $v = 0$ *in* $\mathcal{V}$;

✓ ❷ $\|\lambda v\| = |\lambda|\,\|v\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $v$ *in* $\mathcal{V}$;

✓ ❸ $\|u+v\| \le \|u\| + \|v\|$ *for all* $u$ *and* $v$ *in* $\mathcal{V}$ *(the triangle inequality).*

Goal to show $\|v\|_2$ satisfies these properties...

$$\|v\|_2 = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$$

✓ ① if $v=0$ then $\|v_2\| = \sqrt{|0|^2 + |0|^2 + \cdots + |0|^2} = 0$

on the other hand if

$$\sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2} = 0$$ then each $v_i = 0$.

② $$\|\lambda v\|_2 = \sqrt{|\lambda v_1|^2 + |\lambda v_2|^2 + \cdots + |\lambda v_n|^2}$$

$$= \sqrt{|\lambda|^2 \left(|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2\right)} = |\lambda|\,\|v\|_2$$

③ Need to show triangle inequality. $\|u+v\| \leq \|u\| + \|v\|$

Consider $\|u+v\|^2$ introduce a parameter

$\|\lambda u+v\|^2$ note if $\lambda=0$ then $\|0+v\| \underset{\leq}{=} \|0\| + \|v\|$

View definition algebraically. If $v \in \mathbb{R}^n$

$$\|v\|_2 = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$$

$$\|v\|_2^2 = |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2$$

$$= v_1^2 + v_2^2 + \cdots + v_n^2 = v \cdot v,$$

Thus

$$0 \leq \|\lambda u+v\|_2^2 = (\lambda u+v) \cdot (\lambda u+v)$$

$$= \lambda^2 u \cdot u + \underbrace{\lambda u \cdot v + \lambda v \cdot u} + v \cdot v$$

$$= \underbrace{\lambda^2 \|u\|_2^2 + 2\lambda u \cdot v + \|v\|_2^2}$$

Therefore, no matter what $\lambda$ is this quadratic is non-negative...

$$p(\lambda) = \lambda^2 \|u\|_2^2 + 2\lambda u \cdot v + \|v\|_2^2$$

positive. Concave up.

$p(\lambda) = a\lambda^2 + b\lambda + c$

$a = \|u\|_2^2$

$b = 2u \cdot v$

$c = \|v\|_2^2$

need to make sure the quadratic doesn't cross the $x$-axis...

solve $P(\lambda) = 0$ to check

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac > 0$ then we get two real roots like in the picture

Therefore $b^2 - 4ac \leq 0$ so this doesn't happen.

$$(2u \cdot v)^2 \leq 4 \|u\|_2^2 \|v\|_2^2$$

or

$$\boxed{|u \cdot v| \leq \|u\|_2 \|v\|_2}$$

Cauchy inequality...

Plug this inequality back in set $\lambda = 1$

$$0 \leq \| u+v \|_2^2 = (u+v)\cdot(u+v)$$

$$= u \cdot u + \underbrace{u\cdot v + v \cdot u} + v \cdot v$$

$$= \|u\|_2^2 + 2\, u\cdot v + \|v\|_2^2$$

$$\leq \|u\|_2^2 + |2\, u\cdot v| + \|v\|_2^2$$

now factor

$$\leq \|u\|_2^2 + 2\|u\|_2\|v\|_2 + \|v\|_2^2$$

$$= (\|u\|_2 + \|v\|_2)^2$$

Thus

$$\| u+v \|_2^2 \leq (\|u\|_2 + \|v\|_2)^2$$

or

$$\| u+v \|_2 \leq \|u\|_2 + \|v\|_2$$ triangle inequality

Do the same thing for any $p \in (1,2) \cup (2,\infty)$

$$\|r\|_p = \sqrt[p]{\sum_{i=1}^{n} |r_i|^p}$$

already did $p=1$ and $p=2$

Triangle inequality is the only difficulty...

First idea prove that

Young's inequality

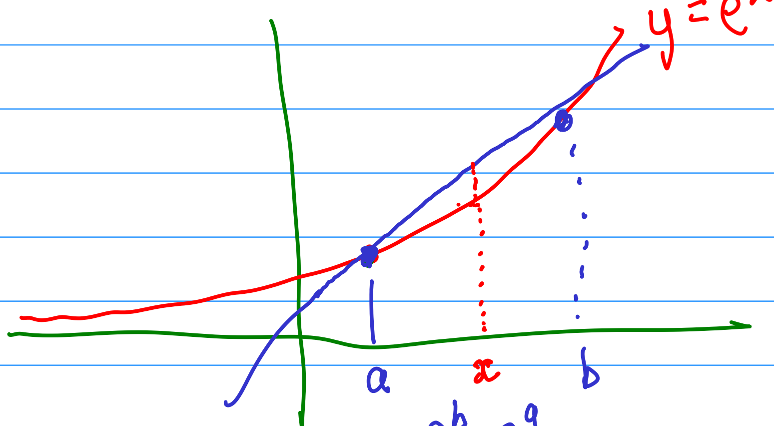
$$\boxed{ab \leq \frac{a^p}{p} + \frac{a^q}{q}}$$

when $p \in (1, \infty)$ and $q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Solve for $q$:

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \quad \text{or} \quad q = \frac{p}{p-1}$$

Proof in the book used convexity.




$y = e^x$ is convex

$$\ell(x) = e^a + \frac{e^b - e^a}{b-a}(x-a)$$

$$\ell(x) \geq e^x$$

since $x \in (a,b)$ there is $\theta \in (0,1)$ such that

$$x = \theta a + (1-\theta) b$$

Therefore $\ell(\vartheta a+(1-\vartheta)b) \geq e^{\vartheta a+(1-\vartheta)b}$

$$\ell(x) = e^a + \frac{e^b-e^a}{b-a}(x-a) = e^a + \frac{e^b-e^a}{b-a}(\vartheta a+(1-\vartheta)b-a)$$

$$= \frac{e^a(b-a) + (e^b-e^a)(\vartheta a+(1-\vartheta)b-a)}{b-a}$$

$$= \frac{e^a(b-a-\vartheta a-(1-\vartheta)b+a)+e^b(\vartheta a+(1-\vartheta)b-a)}{b-a}$$

$$= \frac{e^a(\vartheta b-(\vartheta+1)a+a) + e^b((\vartheta-1)a+(1-\vartheta)b)}{b-a}$$

$$= \frac{e^a(\vartheta(b-a)) + e^b((1-\vartheta)(b-a))}{b-a}$$

$$\boxed{= \vartheta e^a+(1-\vartheta)e^b \geq \boxed{e^{\vartheta a+(1-\vartheta)b}}}$$

Want to prove

$$ab \leq \frac{a^p}{p} + \frac{a^q}{q} \quad \text{where} \quad \frac{1}{p}+\frac{1}{q} = 1$$

$\vartheta = \frac{1}{p}$ $\quad 1-\vartheta = \frac{1}{q}$

$$a = e^{\ln a} = e^{\frac{1}{p}\ln a^p}$$

$$b = e^{\ln b} = e^{\frac{1}{q}\ln b^q}$$

$$ab = e^{\frac{1}{p}\ln a^p} e^{\frac{1}{q}\ln b^q} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q}$$

$$= \boxed{e^{\theta \ln a^p + (1-\theta)\ln b^q}} \leq \theta e^{\ln a^p} + (1-\theta) e^{\ln b^q}$$

$$= \theta a^p + (1-\theta) b^q = \frac{a^p}{p} + \frac{b^q}{q}$$

Therefore $\quad ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q} \quad$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$

remark $a>0$ and $b>0$ was also assumed.

✓ **Theorem 2.5 (Hölder's inequality)** *Let $p, q > 1$, $(1/p) + (1/q) = 1$. Then, for any $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^n$, we have*

$$\left| \sum_{i=1}^{n} u_i v_i \right| \leq \|\boldsymbol{u}\|_p \|\boldsymbol{v}\|_q \, .$$

Idea is to normalize the u and v with respect to the p and q norms

$$\tilde{u} = \frac{u}{\|u\|_p} \qquad \tilde{v} = \frac{v}{\|v\|_q}$$

Young's ineq..

$$\left|\sum_{i=1}^{n} \tilde{u}_i \tilde{v}_i\right| \le \sum_{i=1}^{n} |\tilde{u}_i||\tilde{v}_i|$$

$$\le \sum_{i=1}^{n} \left( \frac{|\tilde{u}_i|^p}{p} + \frac{|\tilde{v}_i|^q}{q} \right)$$

$$\le \frac{1}{p} \boxed{\sum_{i=1}^{n} |\tilde{u}_i|^p} + \frac{1}{q} \boxed{\sum_{i=1}^{n} |\tilde{v}_i|^q}$$

$$= \frac{1}{p} \|\tilde{u}\|_p^p + \frac{1}{q} \|\tilde{v}\|_q^q = \frac{1}{p} + \frac{1}{q} = 1$$

equal to 1 since normalized

Therefore $\left|\sum_{i=1}^{n} \tilde{u}_i \tilde{v}_i\right| \le 1$

plug in $\tilde{u} = \dfrac{u}{\|u\|_p}$ $\qquad \tilde{v} = \dfrac{v}{\|v\|_q}$

$$\left|\sum_{i=1}^{n} \frac{u_i}{\|u\|_p} \frac{v_i}{\|v\|_q}\right| \le 1$$

multiply

and so.

$$\left|\sum u_i v_i\right| \le \|u\|_p \|v\|_q$$

Over the weekend look at ...

**Theorem 2.6** (**Minkowski's inequality**) *Let* $1 \leq p \leq \infty$ *and* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. *Then,*

$$\|\boldsymbol{u}+\boldsymbol{v}\|_p \leq \|\boldsymbol{u}\|_p + \|\boldsymbol{v}\|_p .$$

triangle inequality for p norms...